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INTRODUCTORY TREATISE

ON

RIGID DYNAMICS.



BY

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PREFACE.

It is hoped that the present work may meet the requirements of two classes of students. It is intended on the one hand to supply a clear account, complete as far as it goes, of the principles on which the calculation of the motion of a rigid body is conducted, for students who have not sufficient time to master the larger treatises already published on the subject. In addition, the writer's experience has shewn him that, in the case of students of a different class, the study of such a work as the present often forms a good preparation for an acquaintance with works which take a wider and deeper range.

The chief difficulty experienced has been that of selection, from a large mass of propositions, of those which were proper to be included as ~~of most~~ importance. Stress has mainly been laid upon the clear enunciation of general principles, without a full perception of which there is sure to be confusion: while less importance has been attached to special artifices for the solution of particular problems.

The author desires to express his thanks to several friends, and especially to his colleague Professor A. S. Herschel, for assistance in correcting proofs and for many valuable hints: and will be grateful to any of his readers who will inform him of errors, or suggest improvements.

COLLEGE OF SCIENCE,

NEWCASTLE-UPON-TYNE,

March, 1882.

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INTRODUCTORY TREATISE

ON

RIGID DYNAMICS.

CHAPTER I.

KINEMATICS.

1. THE motion of a point in space is determined by means of the changes in the co-ordinates which at any instant determine its position. If its co-ordinates referred to three fixed rectangular axes be x, y, z ; the velocities of the particle resolved parallel to these axes are $\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}$ respectively; and the accelerations in the same directions are $\frac{d^2x}{dt^2}, \frac{d^2y}{dt^2}, \frac{d^2z}{dt^2}$ respectively.

Hence to determine completely the motion of a point in space *three* quantities require to be completely known in terms of the time.

2. If the motion of the particle be confined to one plane, *two* quantities are sufficient. If the position of the particle be determined by rectangular co-ordinates x, y , its velocities parallel to these respectively are $\frac{dx}{dt}, \frac{dy}{dt}$, and its accelerations in the same directions are $\frac{d^2x}{dt^2}, \frac{d^2y}{dt^2}$. If the position be determined by polar co-ordinates r, θ , the velocities of the particle along and perpendicular to the radius vector are

$\frac{dr}{dt}$, $r \frac{d\theta}{dt}$, respectively; and its accelerations in the same directions are

$$\frac{d^2r}{dt^2} - r \left(\frac{d\theta}{dt} \right)^2 \text{ and } \frac{1}{r} \frac{d}{dt} \left(r^2 \frac{d\theta}{dt} \right).$$

If $s = f(\phi)$ be the intrinsic equation of the path of the particle, the velocity is entirely along the tangent, and is measured by $\frac{ds}{dt}$, while the accelerations along the tangent and normal are measured by $\frac{d^2s}{dt^2}$ and $\frac{ds}{dt} \cdot \frac{d\phi}{dt}$, respectively. The latter is usually expressed in the form $\frac{1}{\rho} \left(\frac{ds}{dt} \right)^2$ where ρ is the radius of curvature of the curve at the point considered, and is equal to $\frac{ds}{d\phi}$.

If a point be moving in a circle round the origin so that r is constant and $\frac{dr}{dt}$ therefore vanishes, the velocity of the particle is entirely perpendicular to the radius vector, and is measured by $r \frac{d\theta}{dt}$, or $r\omega$, if ω denote the angular velocity at the instant.

The resolved parts of this velocity parallel to the axes of x and y are $-y\omega$ and $+x\omega$ respectively, if ω denote a rotation from Ox to Oy .

All the preceding results are to be found in the ordinary treatises on Dynamics of a particle.

3. The motion of a straight rod is completely determined if we know at any time the position of a given point on the rod, and the direction of the line of the rod in relation to certain fixed directions. If the motion be confined to one plane, this will require the determination of three things, viz. the co-ordinates x, y of the fixed point, and the inclination θ of the line to Ox , in terms of t , the time. The rate of change of this latter quantity, measured by $\frac{d\theta}{dt}$, is called the angular velocity of the line in the plane.

It is obvious that the motion of a plane figure of any kind in its own plane, is completely determined if we know that of any one line in it, and therefore requires *three* quantities for its determination.

If u, v be the velocities of a given point in the plane figure parallel to the axes, and x, y be the co-ordinates of any point in the figure relative to this point as origin, the velocities of the second point parallel to the axes will be $u - y\omega, v + x\omega$, where ω is the angular velocity of the body round the given point.

If a point be found such that these expressions vanish, that point is instantaneously at rest. Its co-ordinates are $-\frac{v}{\omega}, \frac{u}{\omega}$, and if these be called α, β , the velocities of any other point parallel to the axes are, by substitution of $\beta\omega$ and $-\alpha\omega$ for u and v , $-(y - \beta)\omega$ and $(x - \alpha)\omega$. These expressions shew that the motion is instantaneously one of rotation about the point (α, β) which is therefore called the centre of instantaneous rotation.

The position of this point can be geometrically determined if we know the directions of motion of any two points of the body, for it will evidently be the point of intersection of the lines drawn from these points in directions at right angles to those of their motion.

4. If the motion of the rod be not confined to one plane, three quantities will determine the motion of the fixed point, and three more the direction of the line relative to the co-ordinate axes. These latter three are however connected by the well-known relation between the direction-cosines of any straight line, and consequently the total number of quantities to be found is five.

5. It is evident that the position of a rigid body of any given form is completely known if we know the positions of any three points of the body which do not lie in a straight line. This will require the determination of the nine co-ordinates of the three points. As however these nine co-ordinates are connected by three relations, obtained from the consideration that the distances between

the points are given, there remain *six* independent quantities. Thus six conditions are required in order to determine completely the most general motion of a single rigid body.

6. Let (x, y, z) be the co-ordinates of any point of the body; (α, β, γ) those of another point on the body, and ξ, η, ζ the co-ordinates of the first point relative to axes through the second parallel to the original co-ordinate axes.

$$\text{Therefore } x = \alpha + \xi, \quad y = \beta + \eta, \quad z = \gamma + \zeta;$$

$$\text{Hence } \frac{dx}{dt} = \frac{d\alpha}{dt} + \frac{d\xi}{dt}, \quad \frac{dy}{dt} = \frac{d\beta}{dt} + \frac{d\eta}{dt}, \quad \frac{dz}{dt} = \frac{d\gamma}{dt} + \frac{d\zeta}{dt}.$$

Hence the velocity of (x, y, z) in any direction is equal to the sum of the velocities of (α, β, γ) , and of (x, y, z) relative to (α, β, γ) , in that direction.

To investigate the motion of a rigid body, we may therefore investigate the motion of any given point in it, and add the motion of the body relative to that point.

7. Let then (α, β, γ) be taken as origin, and let x', y', z' , be the co-ordinates of the point (ξ, η, ζ) referred to axes fixed in the body and moving with it.

Let $(l_1, m_1, n_1), (l_2, m_2, n_2), (l_3, m_3, n_3)$ be the direction-cosines of the axes of x', y', z' with respect to those of ξ, η, ζ . Then by the author's Solid Geometry, Art. 44,

$$\xi = l_1 x' + l_2 y' + l_3 z', \quad x' = l_1 \xi + m_1 \eta + n_1 \zeta,$$

$$\eta = m_1 x' + m_2 y' + m_3 z', \quad y' = l_2 \xi + m_2 \eta + n_2 \zeta,$$

$$\zeta = n_1 x' + n_2 y' + n_3 z', \quad z' = l_3 \xi + m_3 \eta + n_3 \zeta,$$

$$\text{Therefore } \frac{d\xi}{dt} = x' \frac{dl_1}{dt} + y' \frac{dl_2}{dt} + z' \frac{dl_3}{dt}$$

$$= \xi \left(l_1 \frac{dl_1}{dt} + l_2 \frac{dl_2}{dt} + l_3 \frac{dl_3}{dt} \right) + \eta \left(m_1 \frac{dl_1}{dt} + m_2 \frac{dl_2}{dt} + m_3 \frac{dl_3}{dt} \right) + \zeta \left(n_1 \frac{dl_1}{dt} + n_2 \frac{dl_2}{dt} + n_3 \frac{dl_3}{dt} \right).$$

But (Solid Geometry, Art. 44) we have

$$\begin{aligned} l_1^2 + l_2^2 + l_3^2 &= 1 \dots (1) & m_1 n_1 + m_2 n_2 + m_3 n_3 &= 0 \dots (4) \\ m_1^2 + m_2^2 + m_3^2 &= 1 \dots (2) & n_1 l_1 + n_2 l_2 + n_3 l_3 &= 0 \dots (5) \\ n_1^2 + n_2^2 + n_3^2 &= 1 \dots (3) & l_1 m_1 + l_2 m_2 + l_3 m_3 &= 0 \dots (6). \end{aligned}$$

Differentiating these equations we get

$$l_1 \frac{dl_1}{dt} + l_2 \frac{dl_2}{dt} + l_3 \frac{dl_3}{dt} = 0,$$

and two similar relations from (2) and (3); and also

$$m_1 \frac{dn_1}{dt} + m_2 \frac{dn_2}{dt} + m_3 \frac{dn_3}{dt} = - \left(n_1 \frac{dm_1}{dt} + n_2 \frac{dm_2}{dt} + n_3 \frac{dm_3}{dt} \right) \dots (7),$$

with two similar relations from (5) and (6).

If we agree to denote the value of each side of the equation (7) by the symbol ω_x , and denote the corresponding expressions $n_1 \frac{dl_1}{dt} + n_2 \frac{dl_2}{dt} + n_3 \frac{dl_3}{dt}$, $l_1 \frac{dm_1}{dt} + l_2 \frac{dm_2}{dt} + l_3 \frac{dm_3}{dt}$, by the symbols ω_y , ω_z , we shall obtain

$$\left. \begin{aligned} \frac{d\xi}{dt} &= \omega_y \xi - \omega_z \eta \\ \frac{d\eta}{dt} &= \omega_z \xi - \omega_x \zeta \\ \frac{d\zeta}{dt} &= \omega_x \eta - \omega_y \xi \end{aligned} \right\} \dots \dots \dots (8).$$

and similarly

The three quantities ω_x , ω_y , ω_z evidently depend on the change of position of the axes fixed in the body, relatively to axes fixed in space, that is on the motion of the body considered as a whole. We shall be able presently to give them a more definite geometrical meaning. The equations (8) thus determine the velocity of each particle in terms of the general motion of the body.

8. If u , v , w be the velocities of the point (α, β, γ) parallel to the co-ordinate axes, the whole velocities of the point (x, y, z) are obtained by adding u , v , w to the values of $\frac{d\xi}{dt}$, $\frac{d\eta}{dt}$, $\frac{d\zeta}{dt}$ respectively; thus we have

$$\frac{dx}{dt} = u + \omega_y(z - \gamma) - \omega_z(y - \beta),$$

$$\frac{dy}{dt} = v + \omega_z(x - \alpha) - \omega_x(z - \gamma),$$

$$\frac{dz}{dt} = w + \omega_x(y - \beta) - \omega_y(x - \alpha).$$

The motion of every point of the body is therefore completely determined if $u, v, w, \omega_x, \omega_y, \omega_z$ be known.

9. Returning now to the consideration and interpretation of the equations (8) of Article 7, let us suppose a straight line through the point (α, β, γ) of the body taken as origin, whose equations are

$$\frac{x}{\omega_x} = \frac{y}{\omega_y} = \frac{z}{\omega_z} \dots\dots\dots(1).$$

The square of the length of the perpendicular on this line from a point (ξ, η, ζ) is (Solid Geometry, Art. 28,)

$$\begin{aligned} \xi^2 + \eta^2 + \zeta^2 - \frac{(\xi\omega_x + \eta\omega_y + \zeta\omega_z)^2}{\omega_x^2 + \omega_y^2 + \omega_z^2} \\ = \frac{(\omega_y\zeta - \omega_z\eta)^2 + (\omega_z\xi - \omega_x\zeta)^2 + (\omega_x\eta - \omega_y\xi)^2}{\omega_x^2 + \omega_y^2 + \omega_z^2}, \end{aligned}$$

or if V represent the resultant velocity of the particle at (ξ, η, ζ) , and we represent $\omega_x^2 + \omega_y^2 + \omega_z^2$ by Ω^2 the square of the perpendicular becomes $\frac{V^2}{\Omega^2}$.

If we call this perpendicular p , we have therefore

$$V = p\Omega \dots\dots\dots(2).$$

Again the direction-cosines of the line of motion of the particle (ξ, η, ζ) are proportional to $\frac{d\xi}{dt}, \frac{d\eta}{dt}, \frac{d\zeta}{dt}$.

From equations (8) of Art. 7 we can deduce the two following relations :

$$\begin{aligned} \xi \frac{d\xi}{dt} + \eta \frac{d\eta}{dt} + \zeta \frac{d\zeta}{dt} = 0, \\ \omega_x \frac{d\xi}{dt} + \omega_y \frac{d\eta}{dt} + \omega_z \frac{d\zeta}{dt} = 0, \end{aligned}$$

whence it follows that the line of motion of the particle (ξ, η, ζ) is perpendicular to the line $\frac{x}{\xi} = \frac{y}{\eta} = \frac{z}{\zeta}$ and also to the line (1), and therefore is perpendicular to the plane containing them both, that is to the plane passing through the line (1) and the point (ξ, η, ζ) .

Hence the instantaneous velocity of the point (ξ, η, ζ) is exactly the same as if it were moving in a circle whose radius is p with angular velocity Ω .

The motion of the whole body is therefore at any instant represented by a rotation round the line (1) with an angular velocity Ω .

It is perhaps unnecessary to remark that as $\omega_x, \omega_y, \omega_z$ probably change from instant to instant, both the direction of the line (1) and the angular velocity round it change also.

The line (1) is called *the instantaneous axis* through the point (α, β, γ) .

10. If at any moment it happens that ω_y and ω_z both vanish, the instantaneous axis coincides with the axis of x and the angular velocity round it becomes ω_x . The values of the velocities of the particle (ξ, η, ζ) parallel to the axes of x, y, z become in that case by equations (8) of Art. 7 respectively $0, -\omega_x \zeta$ and $\omega_x \eta$. The velocities produced by an angular velocity ω_y round the axis of y would similarly be $\omega_y \zeta, 0, -\omega_y \xi$ parallel to Ox, Oy, Oz respectively; and those produced by an angular velocity ω_z round the axis of z would similarly be $-\omega_z \eta, \omega_z \xi, 0$.

Comparing these results with equations (8) of Art. 7 we see that the actual velocities of the particle are the algebraical sum of those which would be produced separately by the separate angular velocities $\omega_x, \omega_y, \omega_z$ round the three axes of x, y , and z , which may therefore be regarded as the resolved parts of Ω in the directions of those axes.

11. It will be evident from the results of the last Article that ω_x represents an angular velocity round the axis of x from Oy towards Oz , for, supposing η, ζ to be positive, such a rotation will give a positive linear velocity of the par-

ticle parallel to Oz , and a negative one parallel to Oy . Similarly ω_y denotes a rotation round Oy from Oz to Ox , and ω_z a rotation round Oz from Ox to Oy . All these rotations are estimated as positive when the couple which would tend to produce them would be positive according to the ordinary statical convention.

If then we measure off on any line a length representing the magnitude of the angular velocity round that line, this length will completely represent the rotation in every respect, the axis of rotation, the velocity of rotation, and the sense in which the body rotates round that axis.

12. The result obtained in Art. 10 can now be enunciated in the following manner.

If there be simultaneously impressed on a body angular velocities represented by three straight lines mutually at right angles, the resultant motion is an angular velocity represented by that diagonal of the parallelepiped of which these three lines are edges, which passes through their point of intersection.

13. The proposition of the last Article may be replaced by the following, which may be called "the parallelogram of angular velocities" and may be enunciated thus :

If two angular velocities represented by two straight lines be simultaneously impressed on a body, the resultant motion will be an angular velocity represented by the diagonal of the parallelogram of which those lines are adjacent sides.

The following geometrical proof can be given :

Let OA , OB represent the angular velocities, and let OC be the diagonal of the parallelogram $OACB$.

Let P be any point on the body, and let PQ be drawn perpendicular to the plane $OACB$, to meet it in Q . From Q draw perpendiculars QH , QK , QL on OA , OB and OC . Join PH , PK , PL . These are perpendicular to OA , OB , OC respectively.

Owing to the angular velocity OA , P has a linear velocity $OA \cdot PH$ in the plane PHQ perpendicular to PH . This may be resolved into velocities, $OA \cdot QH$ along PQ and $OA \cdot PQ$ parallel to HQ .

It follows from the analogy of these formulæ with those giving the moments of a system of forces round axes parallel to those of x, y, z (Todhunter's *Analytical Statics*, Chapter VII.) that all the results relating to the central axis and other properties of the system of forces have their analogues in the present subject.

For instance, if there be one point whose instantaneous velocity is zero there will be a straight line which is instantaneously at rest. The condition for this is

$$u\omega_x + v\omega_y + w\omega_z = 0 \dots\dots\dots(2).$$

In this case the motion is, for the instant, one of rotation round an axis whose equations are obtained by equating any two of the quantities in (1) to zero.

Whether the condition (2) be fulfilled or no, the linear velocity of all points in the straight line given by the equations,

$$\frac{u + \omega_y \zeta - \omega_z \eta}{\omega_x} = \frac{v + \omega_z \xi - \omega_x \zeta}{\omega_y} = \frac{w + \omega_x \eta - \omega_y \xi}{\omega_z},$$

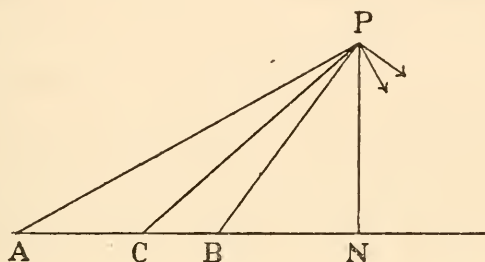
is in the direction of this line and is less than that of all points outside it. The linear velocity of all points on the surface of a right circular cylinder of which this line is the axis and whose radius is c exceeds that of points on the axis by the same quantity, so that if the two velocities be v and v' ,

$$v'^2 = v^2 + c^2(\omega_x^2 + \omega_y^2 + \omega_z^2).$$

16. Any number of simultaneous rotations round intersecting axes can thus be replaced by a rotation round a single axis. Two simultaneous rotations round parallel axes will in general be equivalent to a rotation round a single axis parallel to them. The following geometrical proof may be given.

The motion of each particle is evidently in a plane perpendicular to the two axes. Let P be any particle, let the plane of the paper be the plane of motion and A, B the points in which the two axes of rotation cut this plane. Let ω_1, ω_2 be the angular velocities round these axes. Then the rotation round A produces in P a linear velocity $AP \cdot \omega_1$ perpendicular to AP ; the rotation round B similarly produces a velocity $BP \cdot \omega_2$ perpendicular to PB . If PN be drawn from P perpendicular on AB , these two velocities will give a

component $(\omega_1 + \omega_2) \cdot PN$ parallel to AB and $\omega_1 \cdot AN + \omega_2 \cdot BN$ parallel to PN . If we take a point C between A and B , such that $AC \cdot \omega_1 = BC \cdot \omega_2$ the latter component will be replaced by $CN \cdot (\omega_1 + \omega_2)$. Hence the resultant velocity of P is $CP \cdot (\omega_1 + \omega_2)$ perpendicular to CP . That is, the motion of P is exactly that which would be produced by an angular velocity $(\omega_1 + \omega_2)$ round an axis through C parallel to either of the original ones.



If the two angular velocities be of opposite signs, the resultant angular velocity will be the difference of the two original ones, and the point C will be in the straight line AB produced, at a point such that $AC \cdot \omega_1 = BC \cdot \omega_2$.

17. If the two opposite angular velocities are equal, the linear velocity of P parallel to AB , which is expressed by $(\omega_1 - \omega_2) PN$, vanishes. The linear velocity perpendicular to AB is $\omega_1 \cdot AN - \omega_1 \cdot BN$ or $\omega_1 \cdot AB$. As this is the same for every particle, the motion of the body is in that case one of simple translation perpendicular to AB .

We may again call attention to the analogy of these results with those with which the student is already familiar in Statics, angular velocities corresponding to forces, and velocities of translation to couples.

18. The accelerations of any particle of the body parallel to the axes can be obtained by differentiating the expressions (1) of Art. 15 with respect to t and using the equations (8) of Art. 7. The acceleration parallel to Ox is thus

$$\begin{aligned} &= \frac{du}{dt} + \omega_y \frac{d\zeta}{dt} + \frac{d\omega_y}{dt} \cdot \zeta - \omega_z \frac{d\eta}{dt} - \eta \cdot \frac{d\omega_z}{dt}, \\ &= \frac{du}{dt} + \omega_y (\omega_x \eta - \omega_y \xi) - \omega_z (\omega_z \xi - \omega_x \zeta) + \zeta \frac{d\omega_y}{dt} - \eta \frac{d\omega_z}{dt}, \end{aligned}$$

$$\begin{aligned}
&= \frac{du}{dt} + \omega_x (\omega_y \eta + \omega_z \zeta) - (\omega_y^2 + \omega_z^2) \xi + \zeta \frac{d\omega_y}{dt} - \eta \frac{d\omega_z}{dt}, \\
&= \frac{du}{dt} + p\omega_x - \Omega^2 \xi + \zeta \frac{d\omega_y}{dt} - \eta \frac{d\omega_z}{dt},
\end{aligned}$$

where

$$\begin{aligned}
p &\equiv \omega_x \xi + \omega_y \eta + \omega_z \zeta \\
\Omega^2 &\equiv \omega_x^2 + \omega_y^2 + \omega_z^2.
\end{aligned}$$

The accelerations parallel to Oy and Oz can be similarly deduced.

19. We have now seen that the velocities and accelerations of every point of a rigid body can be determined if we know u , v , w , ω_x , ω_y , ω_z . The determination of these six quantities, when the external forces which act on the different particles of the body are known, is the problem which we have to cope with in the remaining chapters of this book.

EXAMPLES. CHAPTER I.

1. A body has angular velocities represented by ω , -2ω , 3ω round the three co-ordinate axes; find the resultant angular velocity.

2. Show that an angular velocity ω round any axis may be replaced by an equal angular velocity round any parallel axis at a distance p from the former, together with a motion of translation, of magnitude $p\omega$, perpendicular to the plane containing the two axes.

3. Explain what is meant by a couple of rotatory motion; show that such is equivalent to a single motion of translation. Hence show that the motion of a rigid body may be represented in an infinite number of ways by rotations about two axes.

4. Show that every motion of a rigid body can be represented by a rotation round some axis and a motion of translation along the same axis.

5. A lamina moves in its own plane so that a point fixed in it lies on a straight line fixed in the plane, and that a straight line fixed in it always passes through a point fixed in the plane; the distances from each point to each line being equal. Prove that the motion may be completely represented by a parabola fixed in the lamina rolling upon a parabola fixed in the plane.

6. A straight rod moves in any manner in a plane; prove that, at any instant, the directions of motion of all its particles are tangents to a parabola.

7. A cube has equal angular velocities imparted to it about three edges mutually at right angles which do not meet. Find the resultant velocity of its centre, and show whether the motion is capable of being represented by a single rotation.

8. If the angular velocities, at any time t , about the axes of x, y, z , are proportional respectively to $\cot(m-n)t$, $\cot(n-l)t$ and $\cot(l-m)t$, determine the locus of the instantaneous axis.

9. A rod moves with its extremities on two intersecting lines. Find the direction of motion of any point. If the two lines do not intersect but are at right angles to each other, examine whether the motion can be represented by a single rotation.



CHAPTER II.

D'ALEMBERT'S PRINCIPLE.

20. THE second law of motion tells us that change of motion in a particle is proportional to, and is in the direction of, the impressed force.

If α be the acceleration of a particle resolved in any direction, m the mass of the particle, and P the external force, measured in pounds or any other unit, acting on the particle in the same direction; it only requires the units of mass and force to be properly chosen to give as the result of the second law of motion the equation

$$m\alpha = P.$$

Writing this equation in the form

$$P - m\alpha = 0$$

if we agree to give to $m\alpha$ the name of "the moving, or effective force" on the particle in the direction considered, we can express this result verbally in the statement that supposing a force equal and opposite to the "effective or moving force" were applied to the particle, this would form with the actual impressed forces, a system in equilibrium.

21. The principle commonly known as D'Alembert's Principle extends this theorem to the case of any number of particles mutually acting and reacting, whether they form what is called a rigid body, a flexible or fluid mass, or a discontinuous system of particles.

For any such system we have the law, that the whole set of external impressed forces acting on the body, with a set of fictitious forces equal to the several "moving" or "effective

forces" on the several particles of the system reversed, would form a system of forces in equilibrium.

This statement is equivalent to the assumption which may be regarded as an extension of Newton's third law of motion, that all the mutual actions and reactions between the particles of the body form a system in equilibrium.

For let P be the resultant external force acting on any particle, R the resultant of all the actions of the other particles on this one; m its mass and α its resultant acceleration. Then the resultant of P and R is $m\alpha$, or P , R and $-m\alpha$ will be in equilibrium. This being true for each particle, the whole set of forces P , the whole set of forces R , and the whole set of reversed effective forces are in equilibrium. If then the set of forces R be separately in equilibrium, it follows that the set of forces P would be in equilibrium with the set of all the counter-effective, or effective forces.

22. The student must bear in mind that the reversed, or counter-effective forces are not actually existent but merely hypothetical quantities which, if introduced, would produce equilibrium. Thus every problem of motion is reduced to one of equilibrium between actual and hypothetical forces.

23. We are now able to write down the equations of motion of any system of particles, provided we know the forces which act on each of them.

Let x, y, z be the co-ordinates of any particle of mass m ; the effective forces on this particle parallel to the axes of x, y, z are $m \frac{d^2x}{dt^2}$, $m \frac{d^2y}{dt^2}$, $m \frac{d^2z}{dt^2}$ respectively.

Consequently, if mX, mY, mZ represent the components of the external force acting on the same particle in the same directions, the system of forces

$$m \left(X - \frac{d^2x}{dt^2} \right), m \left(Y - \frac{d^2y}{dt^2} \right), m \left(Z - \frac{d^2z}{dt^2} \right)$$

acting at (x, y, z) ; and similar forces acting on the other particles of the body form a system in equilibrium.

The six conditions for this are (Todhunter's *Analytical Statics*, Art. 73),

$$\begin{aligned}\Sigma m \left(X - \frac{d^2x}{dt^2} \right) &= 0; \quad \Sigma m \left(Y - \frac{d^2y}{dt^2} \right) = 0; \quad \Sigma m \left(Z - \frac{d^2z}{dt^2} \right) = 0, \\ \Sigma m \left\{ \left(Z - \frac{d^2z}{dt^2} \right) y - \left(Y - \frac{d^2y}{dt^2} \right) z \right\} &= 0, \\ \Sigma m \left\{ \left(X - \frac{d^2x}{dt^2} \right) z - \left(Z - \frac{d^2z}{dt^2} \right) x \right\} &= 0, \\ \Sigma m \left\{ \left(Y - \frac{d^2y}{dt^2} \right) x - \left(X - \frac{d^2x}{dt^2} \right) y \right\} &= 0,\end{aligned}$$

where the symbol Σ indicates summation for all the different particles of the body considered, and will therefore in the case of a continuous body represent a process of integration.

These equations can be written

$$\begin{aligned}\Sigma m \frac{d^2x}{dt^2} &= \Sigma m X; \quad \Sigma m \frac{d^2y}{dt^2} = \Sigma m Y; \quad \Sigma m \frac{d^2z}{dt^2} = \Sigma m Z \dots (1), \\ \Sigma m \left\{ y \frac{d^2z}{dt^2} - z \frac{d^2y}{dt^2} \right\} &= L; \quad \Sigma m \left\{ z \frac{d^2x}{dt^2} - x \frac{d^2z}{dt^2} \right\} = M; \\ \Sigma m \left\{ x \frac{d^2y}{dt^2} - y \frac{d^2x}{dt^2} \right\} &= N \dots (2),\end{aligned}$$

where L , M , N represent the moments of the impressed external forces round the axes of x , y , and z respectively.

24. The solution of these equations can only be attempted when the accelerations of the separate particles have been expressed in terms of the six quantities by which we have seen that the motion of the body as a whole is determined. We will however deduce from them in their present shape two very important laws.

(1) *The equations which give the motion of the centre of inertia of any body or system of bodies are the same as if the whole mass considered were collected at the centre of inertia and acted on there by forces equal and parallel to all those which act on the system.*

(2) *The equations which give the motion of a rigid body relative to its centre of inertia are the same as if the latter were a fixed point.*

25. In order to prove the above principles let us suppose \bar{x} , \bar{y} , \bar{z} to be the co-ordinates of the centre of inertia of the

body under consideration. Let x', y', z' be the co-ordinates of the particle (x, y, z) relative to axes through $(\bar{x}, \bar{y}, \bar{z})$ parallel to the original axes,

$$\text{hence} \quad x = \bar{x} + x', \quad y = \bar{y} + y', \quad z = \bar{z} + z',$$

$$\begin{aligned} \text{and} \quad \Sigma m x &= \Sigma m \bar{x} + \Sigma m x', \\ &= \bar{x} \cdot \Sigma m + \Sigma m x'. \end{aligned}$$

But $\bar{x} \Sigma (m) = \Sigma m x$ by a well-known statical theorem, therefore

$$\Sigma m x' = 0.$$

$$\text{Similarly} \quad \Sigma m y' = 0, \quad \Sigma m z' = 0.$$

$$\text{Also} \quad \Sigma m \frac{d^2 x}{dt^2} = \frac{d^2 \bar{x}}{dt^2} \cdot \Sigma m, \quad \Sigma m \frac{d^2 x'}{dt^2} = 0 \dots \dots \dots (3),$$

$$\begin{aligned} \text{whence} \quad & \left. \begin{aligned} \frac{d^2 \bar{x}}{dt^2} \cdot \Sigma m &= \Sigma m X, \\ \frac{d^2 \bar{y}}{dt^2} \cdot \Sigma m &= \Sigma m Y, \\ \frac{d^2 \bar{z}}{dt^2} \cdot \Sigma m &= \Sigma m Z, \end{aligned} \right\} \dots \dots \dots (4), \\ \text{similarly} \quad & \\ \text{and} \quad & \end{aligned}$$

which prove the first principle.

Also the first of equations (2) of Art. 23 becomes

$$\begin{aligned} \Sigma m \left\{ (\bar{y} + y') \left(\frac{d^2 \bar{z}}{dt^2} + \frac{d^2 z'}{dt^2} \right) - (\bar{z} + z') \left(\frac{d^2 \bar{y}}{dt^2} + \frac{d^2 y'}{dt^2} \right) \right\} \\ = \Sigma m \{ (\bar{y} + y') Z - (\bar{z} + z') Y \}, \end{aligned}$$

which by means of the results in (3) and (4) easily gives

$$\Sigma m \left\{ y' \frac{d^2 z'}{dt^2} - z' \frac{d^2 y'}{dt^2} \right\} = \Sigma m (y' Z - z' Y) \dots \dots (5),$$

and this with two similar equations deduced from the other two of (2) in Art. 23 gives the second principle.

26. It follows from equations (4) of the last Article that if a system of particles be acted on only by their mutual actions and reactions, the centre of inertia of the system either is at rest or moves uniformly in a straight line.

For in this case $\Sigma mX = 0$, $\Sigma mY = 0$, $\Sigma mZ = 0$,

therefore
$$\frac{d^2\bar{x}}{dt^2} = 0, \quad \frac{d^2\bar{y}}{dt^2} = 0, \quad \frac{d^2\bar{z}}{dt^2} = 0,$$

and
$$\frac{d\bar{x}}{dt} = a, \quad \frac{d\bar{y}}{dt} = b, \quad \frac{d\bar{z}}{dt} = c,$$

where a, b, c are constants. Hence the velocity of the centre of inertia is constant both in direction and magnitude.

27. In the case of finite forces the equations (1) and (2) of Art. 23 cannot be integrated directly.

If however the forces acting be enormously great but only act for an exceedingly short interval τ , so that the values of x, y, z may be supposed to remain sensibly unaltered during the time τ , we can integrate with respect to t during that interval. This is the case ordinarily known as that of impulsive forces.

28. If mX be the force acting on any particle of mass m for the interval τ , we give to the quantity $\int_0^\tau mXdt$ the name of impulse. It really represents the whole momentum which would be produced in a free mass m by the force X acting for an interval τ . We shall denote this by the symbol X' , and similarly $\int_0^\tau mYdt$, $\int_0^\tau mZdt$ will be denoted by the symbols Y', Z' .

Let u, v, w be the velocities of the particle (x, y, z) parallel to the axes at the beginning of the time τ ; u', v', w' the values of the same quantities at the end of that interval.

Therefore

$$\begin{aligned} \int_0^\tau \frac{d^2x}{dt^2} dt &= u' - u, \\ \int_0^\tau \frac{d^2y}{dt^2} dt &= v' - v, \\ \int_0^\tau \frac{d^2z}{dt^2} dt &= w' - w. \end{aligned}$$

Whence the equations (1) and (2) when integrated give
 $\Sigma m (u' - u) = \Sigma X', \Sigma m (v' - v) = \Sigma Y', \Sigma m (w' - w) = \Sigma Z',$

$$\left. \begin{aligned} \Sigma m \{y(w' - w) - z(v' - v)\} &= \Sigma (Z'y - Y'z) \\ \Sigma m \{z(u' - u) - x(w' - w)\} &= \Sigma (X'z - Zx) \\ \Sigma m \{x(v' - v) - y(u' - u)\} &= \Sigma (Y'x - X'y) \end{aligned} \right\} \dots (1).$$

If $\bar{u}, \bar{v}, \bar{w}; \bar{u}', \bar{v}', \bar{w}'$, be the velocities of the centre of inertia before and after the impulses, we easily get from (3) of Art. 25

$$(\bar{u}' - \bar{u}) \Sigma (m) = \Sigma \{m (u' - u)\},$$

whence $(\bar{u}' - \bar{u}) \Sigma m = \Sigma X', (\bar{v}' - \bar{v}) \Sigma m = \Sigma Y',$

$$(\bar{w}' - \bar{w}) \Sigma m = \Sigma Z' \dots \dots \dots (2).$$

29. We may notice that, whereas in the case of finite forces such as gravity, the symbol Σ on the right-hand side of the equations usually denotes a summation of an indefinite number of indefinitely small terms, in other words an integration, in the case of impulsive forces there is usually only a small number of impulses at definite points to be considered.

We may farther notice that all finite forces may be left out of consideration in calculating the effect of the impulsive forces, since $\int_0^\tau P dt$ will be indefinitely small unless P is indefinitely large, the supposition being that τ is indefinitely small.

EXAMPLES. CHAPTER II.

1. Apply D'Alembert's Principle to the solution of the following problems.

(α) A heavy chain, flexible and inextensible, homogeneous and smooth, hangs over a small pulley at the common vertex of two smooth inclined planes; it is required to determine the motion of the chain.

(β) A straight tube of small bore revolves uniformly in a horizontal plane about a point at a distance (c) from it, and has a smooth chain of length $2a$ within it, with its middle point initially at rest at the shortest distance from the point. Prove that the space described along the tube in time t is

$$\frac{c}{2} (\epsilon^{\omega t} - \epsilon^{-\omega t})$$

and that the tension at any point of the chain is constant throughout the motion.

(γ) A smooth cycloidal tube whose axis is vertical and vertex upwards contains a chain equal to it in length; if the equilibrium of the chain be disturbed, determine the velocity of it in any position.

2. A person standing on perfectly smooth ice falls. His feet are observed to come rapidly forward. In what direction will his head move?

3. What is the effect of the backwards and forwards motion of the rowers in a boat respectively? Explain how it is that by bending rapidly forwards and then slowly backwards without dipping the oars a slight forward motion may be given to the boat.

4. A rod of length $2a$ is suspended from a fixed point by a string of length l attached to one end; if the string and rod revolve about the vertical with uniform angular velocity, their inclinations to the vertical being θ and ϕ respectively, prove that

$$\frac{3l}{a} = \frac{(4 \tan \theta - 3 \tan \phi) \sin \phi}{(\tan \phi - \tan \theta) \sin \theta}.$$



CHAPTER III.

ON MOMENTS AND PRODUCTS OF INERTIA.

30. WE have now obtained equations which completely determine the motion of a rigid body under the action of any forces. It remains to exhibit the methods of solving these equations in different cases.

The most simple case of motion conceivable, if we except that of parallel and equal translation of all the particles of the body, the solution of which only entails by Art. 25, the discussion of the motion of their centre of inertia, is that of rotation round a fixed axis.

If we take this axis as axis of z , and if ω be the angular velocity round it at any instant, we have for the velocities of any particle (x, y, z) parallel to the co-ordinate axes

$$-y\omega, x\omega, 0$$

respectively ((8), Art. 7, or see Art. 2).

The accelerations in the same directions respectively can be obtained by differentiating these velocities with respect to t , and are therefore

$$-y \frac{d\omega}{dt} - \omega \frac{dy}{dt}, \quad x \frac{d\omega}{dt} + \omega \frac{dx}{dt}, \quad 0;$$

or, substituting the above values for $\frac{dx}{dt}$ and $\frac{dy}{dt}$,

$$-y \frac{d\omega}{dt} - x\omega^2, \quad x \frac{d\omega}{dt} - y\omega^2, \quad 0.$$

If these values be substituted for $\frac{d^2x}{dt^2}$, $\frac{d^2y}{dt^2}$, $\frac{d^2z}{dt^2}$,

in equations (2) of Art. (23), we obtain

$$\left. \begin{aligned} -\Sigma mzx \cdot \frac{d\omega}{dt} + \Sigma myz \cdot \omega^2 &= L \\ -\Sigma myz \cdot \frac{d\omega}{dt} - \Sigma mzx \cdot \omega^2 &= M \\ \Sigma m(x^2 + y^2) \frac{d\omega}{dt} &= N \end{aligned} \right\} \dots\dots\dots(1).$$

The quantities ω and $\frac{d\omega}{dt}$ are the same for all the particles of the body and may be taken outside the sign of summation. We see that in this case it is necessary, before proceeding to the actual solution of the dynamical problem, to ascertain the values of the quantities $\Sigma m(x^2 + y^2)$, Σmyz , Σmzx .

In other problems we shall meet with the similar quantities Σmxy , $\Sigma m(y^2 + z^2)$, $\Sigma m(z^2 + x^2)$, and the values of these six sums must be known in order to enable us to determine completely the motion of the body.

31. The three quantities,

$$\Sigma m(y^2 + z^2), \Sigma m(z^2 + x^2), \Sigma m(x^2 + y^2),$$

are called the *moments of inertia* of the body round the axes of x , y and z respectively.

The three quantities Σmyz , Σmzx , Σmxy are called the products of inertia with respect to the same axes.

The quantities Σmx^2 , Σmy^2 , Σmz^2 are sometimes called the moments of inertia of the body with respect to the planes of yz , zx , and xy respectively.

The moment of inertia of a body round any axis may be defined as the sum of the products of the mass of each particle of the body into the square of the distance of that particle from the axis.

The moment of inertia of a body with respect to any plane is the sum of the products of the mass of each particle into the square of its distance from that plane.

32. The moments of inertia of a given body round any two parallel axes or with respect to any two parallel planes are connected by a simple relation.

Let x, y, z be the co-ordinates of the particle of mass m with respect to the original axes. Let $\bar{x}, \bar{y}, \bar{z}$ be the co-ordinates of G the centre of inertia of the whole mass, and x', y', z' the co-ordinates of m with reference to axes through G parallel to the original axes.

Then $x = \bar{x} + x', y = \bar{y} + y', z = \bar{z} + z'.$

$$\begin{aligned}\text{Therefore } \Sigma m(y^2 + z^2) &= \Sigma m(\bar{y} + y')^2 + \Sigma m(\bar{z} + z')^2 \\ &= (\bar{y}^2 + \bar{z}^2) \Sigma m + 2\bar{y} \Sigma m y' + 2\bar{z} \Sigma m z' + \Sigma m(y'^2 + z'^2) \\ &= (\bar{y}^2 + \bar{z}^2) \Sigma m + \Sigma m(y'^2 + z'^2),\end{aligned}$$

since $\Sigma m y' = 0, \Sigma m z' = 0$ by the properties of the centre of inertia.

That is, the moment of inertia about the axis of z is equal to the moment of inertia about a parallel axis through the centre of inertia added to the product of the whole mass into the square of the distance between the two axes.

Similarly we find that

$$\begin{aligned}\Sigma m x^2 &= \bar{x}^2 \cdot \Sigma m + \Sigma m x'^2, \\ \Sigma m y z &= \bar{y} \bar{z} \cdot \Sigma m + \Sigma m y' z'.\end{aligned}$$

33. If therefore the moments and products of inertia of a body with reference to any axes through its centre of inertia be known, those with reference to any parallel axes can be determined.

34. If $\frac{x}{\lambda} = \frac{y}{\mu} = \frac{z}{\nu}$ (1) be the equations of any straight line through the origin, the square of the perpendicular from the point (x, y, z) on this line is known to be equal to

$$x^2 + y^2 + z^2 - \frac{(\lambda x + \mu y + \nu z)^2}{\lambda^2 + \mu^2 + \nu^2},$$

which, if λ, μ, ν be assumed to be the direction-cosines of (1) so that $\lambda^2 + \mu^2 + \nu^2 = 1$, can be written

$$\lambda^2 (y^2 + z^2) + \mu^2 (z^2 + x^2) + \nu^2 (x^2 + y^2) - 2\mu\nu yz - 2\nu\lambda zx - 2\lambda\mu xy.$$

Hence, if I denote the moment of inertia of the body about the line (1),

$$I = \lambda^2 \sum m (y^2 + z^2) + \mu^2 \sum m (z^2 + x^2) + \nu^2 \sum m (x^2 + y^2) \\ - 2\mu\nu \sum m yz - 2\nu\lambda \sum m zx - 2\lambda\mu \sum m xy \\ = A\lambda^2 + B\mu^2 + C\nu^2 - 2A'\mu\nu - 2B'\nu\lambda - 2C'\lambda\mu \dots\dots\dots(2),$$

if A, B, C be the moments of inertia and A', B', C' the products of inertia of the body with reference to the axes of co-ordinates.

35. The expression for I is susceptible of a very simple geometrical interpretation. Let the quadric whose equation

$$\text{is} \quad Ax^2 + By^2 + Cz^2 - 2A'yz - 2B'zx - 2C'xy = \epsilon^4 \dots(3)$$

be constructed. Then, if r be the central radius of this quadric in the direction of the line (1) of the last Article,

$$(A\lambda^2 + B\mu^2 + C\nu^2 - 2A'\mu\nu - 2B'\nu\lambda - 2C'\lambda\mu) r^2 = \epsilon^4,$$

therefore
$$I = \frac{\epsilon^4}{r^2}.$$

That is the moment of inertia of the body round any axis through the origin is inversely proportional to the square of the radius of (3) in the direction of that axis.

Since the moment of inertia round every axis must be a positive quantity it follows that r^2 can never become negative or infinite. Hence the quadric (3) must be an ellipsoid. It is usually called the *momental ellipsoid* of the body with reference to the given origin.

36. We know (Solid Geometry, Arts. 51, 85), that the equation of the quadric can be reduced by transformation of co-ordinates so as to assume the form

$$Px^2 + Qy^2 + Rz^2 = \epsilon^4 \dots\dots\dots(4),$$

where P, Q, R are the roots of the cubic equation

$$(s - A)(s - B)(s - C) - A'^2(s - A) - B'^2(s - B) - C'^2(s - C) \\ + 2A'B'C' = 0 \dots\dots(5).$$

If the equation of the momental ellipsoid were originally calculated with reference to these new axes, the coefficients

of x^2 , y^2 and z^2 would have been the moments of inertia round the axes of x , y , z , and the coefficients of yz , zx , xy would be the products of inertia with reference to the same axes.

Hence the new set of axes have this property, that the products of inertia of the body with reference to them, all vanish: and we see that at every point of a body there must be a set of axes for which this condition is satisfied.

These axes are called *the principal axes* of the body at the given point. Their directions can be determined by equations (3) of Art. 83 of the author's Solid Geometry, merely altering the signs of A' , B' , and C' . The moments of inertia round these axes, which are called the *principal moments of inertia* for the origin, are the roots of equation (5) above.

If the values of P , Q , R be equal, the momental ellipsoid becomes a sphere. Hence since the form of its equation is unaltered by any rotation of the axes, all axes through the given point are principal axes, and the products of inertia will vanish with respect to any such axes whatever.

37. If P , Q , R be the principal moments of inertia of the body at the centre of inertia, the moments and products of inertia with reference to parallel axes through any point (α, β, γ) will be by Art. 32,

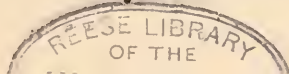
$$P + M(\beta^2 + \gamma^2), \quad Q + M(\gamma^2 + \alpha^2), \quad R + M(\alpha^2 + \beta^2), \\ M\beta\gamma, \quad M\gamma\alpha, \quad M\alpha\beta,$$

where M denotes what we have previously denoted by Σm , the mass of the whole body considered. The \bar{x} , \bar{y} , \bar{z} of Art. 32 are replaced by $-\alpha$, $-\beta$, $-\gamma$ respectively.

Hence the equation of the momental ellipsoid for the point (α, β, γ) is

$$\{P + M(\beta^2 + \gamma^2)\}x^2 + \{Q + M(\gamma^2 + \alpha^2)\}y^2 + \{R + M(\alpha^2 + \beta^2)\}z^2 \\ - 2M\beta\gamma yz - 2M\gamma\alpha zx - 2M\alpha\beta xy = \epsilon^4 \dots (6).$$

Hence the new axes are not principal axes at the new origin unless two of the three quantities α , β , γ vanish. The directions of the principal axes of (6) can however be ascertained by the ordinary methods of Solid Geometry.



38. A given set of co-ordinate axes will only be the principal axes at the origin when all three of the quantities Σmyz , Σmzx , Σmxy vanish. If however two of these quantities, as Σmyz , Σmzx , vanish, one of the co-ordinate axes, in this case that of z , will be a principal axis. For the equation of the momental ellipsoid becomes

$$Ax^2 + By^2 + Cz^2 - 2C'xy = \epsilon^4,$$

and by turning the axes of x , y through an angle θ determined from the equation

$$\tan 2\theta = \frac{2C'}{A - B},$$

the term involving xy can be made to disappear and the equation reduced to the form

$$Px^2 + Qy^2 + Cz^2 = \epsilon^4.$$

From this it is obvious that if any plane divide the body symmetrically, any line perpendicular to this plane is a principal axis at the point where it cuts the plane. For if the origin be transferred to this point and the plane be taken as plane of xy , for every particle m with co-ordinates x , y , z there is an equal particle m with co-ordinates x , y , $-z$. Hence Σmzx and Σmyz both vanish.

39. It is easy to deduce the condition that a given line may be a principal axis at some point of its length.

Take the given line as axis of z and let the origin be transferred to a distance h along it. The values of the products of inertia with reference to the new origin are

$$\Sigma my(z - h), \Sigma m(z - h)x, \Sigma mxy.$$

If the axis of z be a principal axis at the new origin, the former two of these must vanish,

$$\begin{aligned} \text{therefore} \quad \Sigma myz &= h\Sigma my, \quad \Sigma mzx = h\Sigma mx, \\ &= h\bar{y}\Sigma m \quad \quad \quad = h\bar{x}\Sigma m, \end{aligned}$$

if \bar{x} , \bar{y} be co-ordinates of the centre of inertia.

Hence no value of h will satisfy these conditions unless

$$\frac{\Sigma myz}{\bar{y}} = \frac{\Sigma mzx}{\bar{x}}.$$

It is evident that if \bar{x}, \bar{y} both vanish and $\Sigma myz, \Sigma mzx$ vanish also, the required conditions are satisfied by all values of h . Hence a line through the centre of inertia which is a principal axis at any one point is a principal axis at all points through which it passes. This result can also be deduced from equation (6) of Art. 37.

40. We have frequently to calculate the moments of inertia of bodies in the form of a plane lamina. If the plane of the lamina be taken as plane of xy , since the value of z will be zero for every point of the lamina, Σmyz and Σmzx will vanish. Hence one principal axis is always at right angles to the plane of the lamina.

If A, B be the moments of inertia about any two axes at right angles in the plane, and C that about the axis perpendicular to it,

$$A = \Sigma my^2, B = \Sigma mx^2, C = \Sigma m(x^2 + y^2),$$

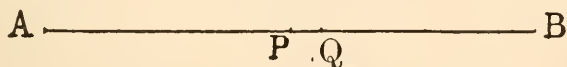
whence

$$C = A + B.$$

41. We have now to determine the moments of inertia in a few simple cases.

A uniform straight rod about an axis through its extremity perpendicular to its length.

Let AB be the rod whose length is a and mass is M , μ the mass of a unit of length, PQ any element of the rod,



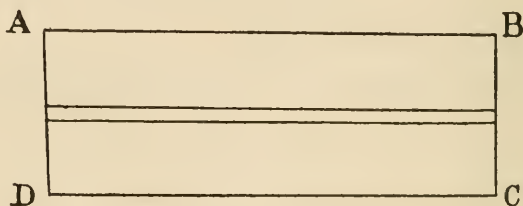
$AP = x, PQ = \delta x$. Then the moment of inertia of the rod about any axis through A perpendicular to AB

$$\begin{aligned} &= \Sigma \{ \mu \delta x \cdot x^2 \} \\ &= \int_0^a \mu x^2 dx \\ &= \frac{1}{3} \mu a^3 = M \cdot \frac{a^2}{3}. \end{aligned}$$

This will also give the moment of inertia of a rod of mass M and length $2a$, about an axis through its middle point perpendicular to its length.

42. *A rectangle about one side.*

Let $ABCD$ be the rectangle whose length AB is a . If we divide the rectangle into an indefinitely large number of indefinitely small strips parallel to AB , the moment of inertia of each strip round AD will be equal to the mass of the strip multiplied by $\frac{a^2}{3}$. Hence if M be the mass of the whole rectangle, the moment of inertia of the rectangle $= M \cdot \frac{a^2}{3}$.



If $AD = b$, the moment of inertia of the rectangle round AB is similarly $= M \cdot \frac{b^2}{3}$.

Hence by Art. 40 the moment of inertia of the rectangle about an axis through A perpendicular to the plane

$$= M \frac{a^2 + b^2}{3}.$$

43. The moment of inertia of a *rectangular parallelepiped* whose edges are a, b, c round the edge c is $M \cdot \frac{a^2 + b^2}{3}$. For the parallelepiped can be divided into any number of thin strips by planes perpendicular to c , the moment of inertia of each of which round c is, by the last Article, equal to its mass $\times \frac{a^2 + b^2}{3}$.

44. The moment of inertia of a *circular ring* whose radius is a , round an axis through its centre perpendicular to its plane is evidently $M \cdot a^2$. Hence, since the moments of

inertia round all diameters must be the same, by Art. 40 it follows that the moment of inertia about a diameter is half the above, or $M \cdot \frac{a^2}{2}$.

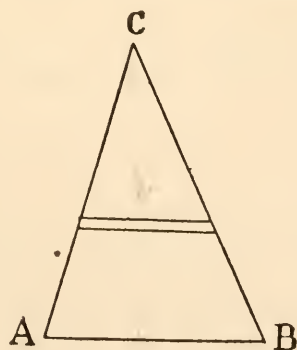
45. *A circular area* of radius a can be divided into a number of concentric rings. The area of one of these may be taken as $2\pi r\delta r$, and if μ be the mass of a unit of area, the moment of inertia of this ring round an axis through the centre perpendicular to its plane is $2\pi\mu r\delta r \cdot r^2$. Hence the moment of inertia of the circular plate

$$\begin{aligned} &= 2\pi\mu \int_0^a r^3 dr \\ &= \frac{1}{2} \pi\mu a^4 = M \cdot \frac{a^2}{2}, \end{aligned}$$

where M is the mass of the plate.

Hence as in Art. 44 the moment of inertia about a diameter is half the above, or $M \cdot \frac{a^2}{4}$.

46. *A triangular area* about its base.

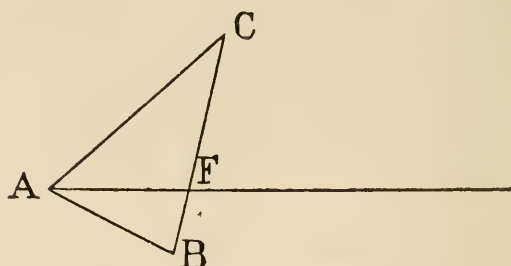


By dividing the area into strips parallel to the base it is easily seen that the moment of inertia

$$= \mu \int_0^h c \cdot \frac{h-z}{h} \cdot z^2 dz,$$

where h is the altitude, c the base of the triangle, and z the distance of any strip from the base. This equals

$$\frac{1}{12} \mu c h^3 = M \cdot \frac{h^2}{6}.$$



The moment of inertia of ABC round any line through A , as AF , is similarly $\frac{1}{12} \mu \cdot AF (h'^3 + h^3)$, where h and h' are the perpendiculars from C and B on AF .

$$\begin{aligned} \text{But this} &= \frac{1}{12} \mu AF (h + h') (h^2 - hh' + h'^2) \\ &= \frac{1}{6} M (h^2 - hh' + h'^2) \\ &= \frac{1}{3} M \left\{ \left(\frac{h}{2} \right)^2 + \left(\frac{h'}{2} \right)^2 + \left(\frac{h - h'}{2} \right)^2 \right\}. \end{aligned}$$

This result is the same as the moment of inertia round AF of three equal particles each of mass $\frac{M}{3}$, placed at the middle points of AC , AB and BC respectively.

Since the centre of inertia of these particles coincides with that of the triangle and the sum of their masses is the same as that of the triangle, it follows by Art. 32 that the moments of inertia of the two systems round any axis parallel to AF , that is round any axis lying in the plane of the triangle, are equal. By Arts. 40 and 34 it follows that the result is true for any axis whatever, and farther that the products of inertia of the two systems with respect to all axes are equal.

47. The moment of inertia of a sphere about a diameter can be determined by dividing the sphere into slices by planes perpendicular to the diameter. The moment of inertia of one of these slices round the required axis by Art. 45

$= \mu\pi (a^2 - x^2) dx \frac{a^2 - x^2}{2}$, where a is the radius of the sphere, x the distance of the slice from the centre, and μ the mass of a unit of volume. Hence the whole moment of inertia required

$$\begin{aligned} &= \frac{1}{2} \mu\pi \int_{-a}^a (a^2 - x^2)^2 dx \\ &= \frac{1}{2} \mu\pi \left[a^4 x - \frac{2a^2 x^3}{3} + \frac{x^5}{5} \right]_{-a}^{+a} \\ &= \frac{8\mu\pi a^5}{15} = M \cdot \frac{2a^2}{5}. \end{aligned}$$

48. The moment of inertia of a spherical shell round a diameter can be obtained independently by integration. It may also be obtained from the formula of the last Article.

The moment of inertia of a sphere of radius $a = \frac{8\mu\pi a^5}{15}$, that of a sphere of radius $a + \delta a$ round the same diameter will be $\frac{8\mu\pi (a + \delta a)^5}{15}$. The difference of these, which neglecting squares and higher powers of δa is $\frac{8\mu\pi a^4 \delta a}{3}$, will be the moment of inertia of the shell of thickness δa . Since the mass of the shell is $4\mu\pi a^2 \delta a$, this result can be expressed in the form $M \cdot \frac{2a^2}{3}$.

49. The foregoing examples are sufficient to indicate the method of calculating moments of inertia. Products of inertia can be similarly calculated, but their determination will usually involve a double integration. Inasmuch however as whenever it is possible we employ principal axes we do not very frequently have occasion for the values of the products of inertia.

50. We may notice that in all cases the moment of inertia has been expressed in the form Mk^2 , that is, as the product of two factors, one being the whole mass and the

other an expression of two dimensions. The square root of the second factor, or k , is called the *radius of gyration* of the body about the given axis. It is in fact the distance from the axis of a particle whose mass and moment of inertia round the axis are equal to those of the given body.

EXAMPLES. CHAPTER III.

1. Find the product of inertia of a uniform right-angled triangle about the two sides containing the right angle.

2. Find the moments of inertia of a uniform ellipse (1) about either of its axes: (2) about a line through its centre perpendicular to its plane.

3. Shew that the moment of inertia of the lemniscate $r^2 = a^2 \cos 2\theta$ about a line in its plane, through its node and perpendicular to its axis is

$$M \frac{3\pi + 8}{48} a^2.$$

4. Find the moments of inertia

(1) Of a portion of the arc of an equiangular spiral about a line through its pole perpendicular to its plane.

(2) Of a parabolic area bounded by the latus rectum about the line joining its vertex to the extremity of its latus rectum.

(3) Of a uniform ellipsoid about a principal axis.

5. Shew that the moment of inertia of a regular polygonal lamina about an axis in its plane through its centre is

$$M \cdot \frac{a^2}{48} \cdot \left\{ 1 + 3 \cot^2 \frac{\pi}{n} \right\},$$

a being the length of a side.

6. The momental ellipse for a uniform triangular lamina is similar and similarly situated to the minimum ellipse circumscribed about the lamina.

7. Find the moment of inertia of a right circular cone whose height is h and semi-vertical angle α , (1) about its axis; (2) about a line through its vertex perpendicular to the axis; (3) about a slant side.

8. Find the moment of inertia of a right circular cone about a generating line, the density of any circular section varying as its distance from the vertex.

9. The axes of an ellipsoid are $2a$, $2b$, $2c$, its density at a point whose distances from the principal planes are x , y , z is

$$\mu \frac{xyz}{abc} :$$

find its moment of inertia about one of the principal axes.

10. Find the moment of inertia of the area of the curve $r = a(1 + \cos \theta)$ about the initial line.

11. Find the moment of inertia of the solid

$$(x^2 + y^2 + z^2 - ax)^2 = a^2(x^2 + y^2 + z^2)$$

about the axis of x .

Find also the moment of inertia of the *surface* of this solid about the same axis.

12. Shew that the moment of inertia of a triangular lamina with respect to any plane is the same as that of three equal particles, each one-third the mass of the triangle, placed at the middle points of the sides, with respect to the same plane.

13. Shew that the moment of inertia of a uniform tetrahedron with respect to any plane through one of its angular points, is the same as that of a system of five particles, one placed at each angular point of the tetrahedron, each of mass one-twentieth that of the tetrahedron, and one of four-fifths the mass of the tetrahedron placed at its centre of inertia, with respect to the same plane.

14. Prove that the result of the last question holds with respect to the moments and products of inertia of the tetrahedron with respect to all axes whatever.

15. If any edge of a uniform tetrahedron be a principal axis, so also is the opposite edge, and the point on either edge at which it is a principal axis will divide the distance between the middle point and the foot of the shortest distance between the edges in the ratio 3 : 2.

16. If M be the mass of a tetrahedron, Q_1 its moment of inertia about any axis through its centre of gravity, Q_2 that of the octahedron formed by joining the middle points of its edges, prove that $Q_1 = \frac{8}{7} Q_2 +$ the moment of inertia of a system of four particles, each of mass $\frac{M}{28}$, placed one at each vertex.

17. Find the principal axes of a right cone, a point on the circumference of the base being the origin; and shew that one of them will pass through the centre of gravity if the angle of the cone be $2 \tan^{-1} \frac{1}{2}$.

18. Shew that two of the principal moments of inertia with respect to a point in a rigid body cannot be equal unless two are equal with respect to the centre of gravity, and the point situated on the axis of unequal moment.

19. If the principal axes at the centre of gravity be taken as the axes of co-ordinates, shew that the locus of points at which the sum of the squares of the three principal moments of inertia $= K^4$ is the surface

$$(x^2 + y^2 + z^2)^2 + (B + C)x^2 + (C + A)y^2 + (A + B)z^2 \\ = \frac{K^4 - A^2 - B^2 - C^2}{2}.$$

20. On a straight cylindrical rod of known length, circular in section and uniform in density, are three anchor rings, A, B, C , different in mass and closely fitted to the rod. The position of A being unaltered, prove that the order of the rings may be changed to A, C, B so as not to affect the centre of gravity of the system of rings and rod, nor the principal moments of inertia of the system at its centre of gravity, nor the distance between B and C .

21. Find the product of inertia of the eighth part of an ellipsoid, cut off by the principal planes, with respect to the axes of x and y .

22. If a body be referred to principal axes through its centre of gravity, and Ma^2 , Mb^2 , Mc^2 be the moments of inertia with respect to them, shew that the moments of inertia with respect to principal axes at a point (α, β, γ) will be given by $M(r^2 + \alpha^2 + \beta^2 + \gamma^2)$, where r^2 is given by the cubic

$$\begin{aligned} (r^2 + \alpha^2 - a^2)(r^2 + \beta^2 - b^2)(r^2 + \gamma^2 - c^2) - (r^2 + \alpha^2 - a^2)\beta^2\gamma^2 \\ - (r^2 + \beta^2 - b^2)\gamma^2\alpha^2 - (r^2 + \gamma^2 - c^2)\alpha^2\beta^2 + 2\alpha^2\beta^2\gamma^2 = 0. \end{aligned}$$

CHAPTER IV.

MOTION ROUND A FIXED AXIS.

51. THE motion of a body may be absolutely free, or it may be restricted by having one point fixed, or two points fixed. If three points be fixed, the body must be at rest. We shall first consider the motion when two points are fixed. It is evident that the motion is one of rotation about the line joining these two points.

52. Let us take one of the fixed points as origin and the straight line joining them as axis of z , and let the distance between them be c . We may assume that the fixed points exercise pressures on the body whose resolved parts parallel to the co-ordinate axes at any instant are respectively $F, G, H; F', G', H'$. As in Art. 30, we have for the values of

$$\frac{d^2x}{dt^2}, \quad \frac{d^2y}{dt^2} \text{ and } \frac{d^2z}{dt^2},$$

to write $-y \frac{d\omega}{dt} - x\omega^2, x \frac{d\omega}{dt} - y\omega^2, 0$

respectively.

With these substitutions the equations (1) and (2) of Art. 23 become

$$-\Sigma my \cdot \frac{d\omega}{dt} - \Sigma mx \cdot \omega^2 = \Sigma mX + F + F' \dots\dots\dots(1)$$

$$\Sigma mx \cdot \frac{d\omega}{dt} - \Sigma my \cdot \omega^2 = \Sigma mY + G + G' \dots\dots\dots(2)$$

$$0 = \Sigma mZ + H + H' \dots\dots\dots(3)$$

$$-\Sigma mzx \cdot \frac{d\omega}{dt} + \Sigma myz\omega^2 = \Sigma m (Zy - Yz) - G'c \dots (4)$$

$$-\Sigma myz \cdot \frac{d\omega}{dt} - \Sigma mzx\omega^2 = \Sigma m (Xz - Zx) + F'c \dots (5)$$

$$\Sigma m (x^2 + y^2) \cdot \frac{d\omega}{dt} = \Sigma m (Yx - Xy) \dots\dots\dots (6)$$

where the symbol Σ on the right-hand side indicates a summation with respect to all the other external forces acting on the body beside the restraint of the axis.

Equation (6) is sufficient to determine ω , if $\Sigma m (Yx - Xy)$ be given; equations (1) and (5) will then determine F and F' , and equations (2) and (4) will give G and G' . Equation (3) gives us $H + H'$, the separate values of H and H' being indeterminate, an obvious consequence of the principle of the transmissibility of force.

53. The most important particular case is that of rotation about a horizontal axis under the action of gravity.

We may take the axis of x to be vertical, and \bar{x} , \bar{y} , \bar{z} as the co-ordinates of the centre of inertia of the body. If θ be the angle which the plane containing the centre of inertia and the axis of z makes with the plane of zx ,

$$\omega = \frac{d\theta}{dt}, \quad \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}.$$

Also $X = g$, $Y = 0$, $Z = 0$, if g be the force of gravity per unit of mass. Let M be the whole mass of the body.

The six equations of the last article become

$$-M\bar{y} \frac{d^2\theta}{dt^2} - M\bar{x} \left(\frac{d\theta}{dt} \right)^2 = Mg + F + F'$$

$$M\bar{x} \frac{d^2\theta}{dt^2} - M\bar{y} \left(\frac{d\theta}{dt} \right)^2 = G + G'$$

$$0 = H + H'$$

$$\begin{aligned}
 -\Sigma mzx \frac{d^2\theta}{dt^2} + \Sigma myz \left(\frac{d\theta}{dt}\right)^2 &= -G'c \\
 -\Sigma myz \frac{d^2\theta}{dt^2} - \Sigma mzx \left(\frac{d\theta}{dt}\right)^2 &= Mg\bar{z} + F'c \\
 \Sigma m(x^2 + y^2) \frac{d^2\theta}{dt^2} &= -Mg\bar{y}.
 \end{aligned}$$

54. Let h be the distance of the centre of inertia from the axis of rotation, and k the *radius of gyration* (Art. 50) round an axis through the centre of inertia parallel to this line, the moment of inertia about the axis of z , or $\Sigma m(x^2 + y^2)$, is therefore $Mk^2 + Mh^2$ (Art. 32). Also $\bar{y} = h \sin \theta$.

The last equation of Art. 52 becomes

$$M(h^2 + k^2) \frac{d^2\theta}{dt^2} = -Mgh \sin \theta,$$

or
$$\frac{d^2\theta}{dt^2} = -\frac{gh}{h^2 + k^2} \cdot \sin \theta \dots\dots\dots(1).$$

This is the equation which determines the motion of a heavy particle suspended by a string of length $\frac{h^2 + k^2}{h}$. This latter quantity is often called the length of the *simple equivalent pendulum*. The time of a small oscillation of such a pendulum of length l is shown in treatises on Dynamics of a particle to be $2\pi \sqrt{\frac{l}{g}}$, or in this case $2\pi \sqrt{\frac{h^2 + k^2}{gh}}$.

By integrating (1) we easily obtain

$$\left(\frac{d\theta}{dt}\right)^2 = C + \frac{2gh}{h^2 + k^2} \cdot \cos \theta \dots\dots\dots(2),$$

where C is a constant to be determined from the initial circumstances of the motion.

From (1) and (2) we can substitute the values of $\frac{d^2\theta}{dt^2}$ and $\left(\frac{d\theta}{dt}\right)^2$ in the equations of the last Article, and obtain the values of $F, F', G, G', H + H'$.

55. If the origin be taken as the point where the perpendicular from the centre of inertia meets the axis of rotation, $\bar{z}=0$. If also the axis of rotation be a principal axis at this point, $\Sigma myz=0$, $\Sigma mzx=0$. This will be the case (Art. 38) if the plane through the centre of inertia perpendicular to the axis divides the body symmetrically.

The equations of Art. 53 then give $F''=0$, $G'=0$,

$$\begin{aligned} -F &= Mg + Mh \cos \theta \left(\frac{d\theta}{dt} \right)^2 + Mh \sin \theta \frac{d^2\theta}{dt^2} \\ &= Mg + MCh \cos \theta + \frac{Mgh^2}{h^2 + k^2} (2 \cos^2 \theta - \sin^2 \theta), \\ -G &= Mh \sin \theta \left(\frac{d\theta}{dt} \right)^2 - Mh \cos \theta \frac{d^2\theta}{dt^2} \\ &= MCh \sin \theta + \frac{3Mgh^2}{h^2 + k^2} \sin \theta \cos \theta, \end{aligned}$$

whence $-F$ and $-G$ which represent the resolved pressures of the body on the axis are known. The whole pressure is of course $\sqrt{F^2 + G^2}$.

56. In the case we have now assumed let O be the origin, G the centre of inertia, and OG consequently perpendicular to the axis of rotation. Produce OG to O' so that

$$\text{O} \quad \text{G} \quad \text{O}'$$

OO' is the length of the simple equivalent pendulum. The point O' is called the centre of oscillation. A heavy particle placed at O' and connected with O by a weightless wire would oscillate round O just as the body does.

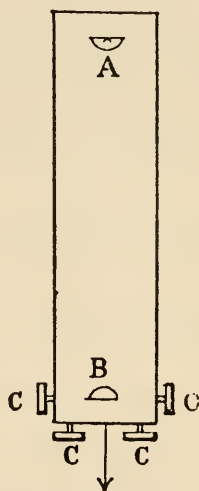
$$\begin{aligned} \text{We have then} \quad OO' &= \frac{h^2 + k^2}{h} \\ &= h + \frac{k^2}{h}, \end{aligned}$$

$$\text{whence} \quad OO' - h \text{ or } O'G = \frac{k^2}{OG},$$

$$\text{or} \quad OG \cdot O'G = k^2.$$

This relation is perfectly symmetrical with respect to O and O' . Hence if the body be suspended from O' , O will be the centre of oscillation, and the times of oscillation will be the same whether the body be suspended from O or O' .

57. This property of the convertibility of the centres of suspension and oscillation is used to determine the length of the simple pendulum which will oscillate in a given time.



A heavy and tolerably uniform bar of metal is pierced with two holes, A and B , chosen so that they shall be as nearly as possible centres of suspension and oscillation to one another. C, C' are heavy screws, by moving which in or out the centre of inertia of the body, and its moment of inertia round an axis through that point can be slightly altered.

The bar is then suspended so that the upper end of the hole A rests on a knife-edge support, and being made to oscillate, the time of oscillation is compared with that of the pendulum of a clock beating in a known time. The time of oscillation when suspended from B is then similarly observed. If these two times are identical, the measurement of AB gives the length of the simple pendulum oscillating in that time. If the times are different, a suitable adjustment of

the screws C, C , will bring them gradually nearer, until finally, when they coincide, AB can be measured. When the length of the simple pendulum for any one time is determined, that for any other time, as for instance a second, can be determined by a proportion.

58. The time of oscillation requires to be observed with very great accuracy. For this purpose it is desirable that the pendulum AB should oscillate nearly in the same time as the clock pendulum. The latter is made to swing just in front of, or just behind AB , so that the clock pendulum and a certain mark attached to AB can be both seen through a telescope suitably placed when at the lowest point of their swing. If they pass the lowest point together in one oscillation, supposing the clock pendulum to go the faster, the next time the latter will pass a little before AB , and the next time it will be still more in advance. Presently, after n oscillations of AB the clock pendulum will very nearly have gained a whole oscillation, and after $n + 1$ of AB will have gained rather more than a whole oscillation, the n th and $n + 1$ th passages of AB being those which most nearly coincide with passages of the clock pendulum. The ratio of the time of oscillation of AB to that of the clock will thus be between $n + 1 : n$ and $n + 2 : n + 1$. If a very large number of oscillations be made and the nearest coincidences noted, we can thus get as close an approximation as we wish to the accurate time of oscillation of AB .

59. If a body rotating about a fixed axis be acted on by sudden impulses, the new angular velocity round the axis and the impulsive strains on the axis can be obtained from the equations of Art. 28. If ω, ω' be the angular velocities before and after the impulsive action, still taking Oz as the axis of rotation, we have as before

$$u = -y\omega, v = x\omega, w = 0; \quad u' = -y\omega', v' = x\omega', w' = 0.$$

If, farther, we take $F_1, G_1, H_1; F'_1, G'_1, H'_1$ to represent the impulses exerted on the body by the axis at two points, one of which is the origin and the other is at a distance c from it; and the symbol Σ denote summation with regard

to all other impulses acting on the body: the equations of Art. 28 become

$$\begin{aligned}
 -(\omega' - \omega) \Sigma mzx &= \Sigma(Z'y - Y'z) - G_1'c, \\
 -(\omega' - \omega) \Sigma myz &= \Sigma(X'z - Z'x) + F_1'c, \\
 (\omega' - \omega) \Sigma m(x^2 + y^2) &= \Sigma(Y'x - X'y), \\
 -M\bar{y}(\omega' - \omega) &= \Sigma X' + F_1' + F_1', \\
 M\bar{x}(\omega' - \omega) &= \Sigma Y' + G_1' + G_1', \\
 0 &= \Sigma Z' + H_1' + H_1'.
 \end{aligned}$$

These equations determine $\omega' - \omega$, F_1 , F_1' , G_1 , G_1' , $H_1 + H_1'$.

60. The most interesting case is when only one external impulse X' , Y' , Z' affects the body, applied at a point (ξ, η, ζ) . In this case it is sometimes possible to apply the impulse without producing any impulsive pressure on the axis of rotation. The point at which the single impulsive force must be applied is called the centre of percussion; more strictly speaking, the line of action of this force ought to be called the line of percussion, since in a really rigid body it is clearly a matter of indifference at what point in its line of action the blow is given.

61. The conditions for the existence of a centre of percussion can be easily discovered.

Assuming that F_1 , F_1' , G_1 , G_1' , H_1 , H_1' all vanish, the equations of the last Article give us

$$\begin{aligned}
 -(\omega' - \omega) \Sigma mzx &= Z'\eta - Y'\zeta, \\
 -(\omega' - \omega) \Sigma myz &= X'\zeta - Z'\xi, \\
 (\omega' - \omega) Mk^2 &= Y'\xi - X'\eta, \\
 -M\bar{y}(\omega' - \omega) &= X', \\
 M\bar{x}(\omega' - \omega) &= Y', \\
 0 &= Z'.
 \end{aligned}$$

From the last three equations we have $\frac{X'}{-\bar{y}} = \frac{Y'}{\bar{x}} = \frac{Z'}{0}$.

Hence the direction of the impulse is perpendicular to the plane containing the centre of inertia and the axis of rotation, the equation of this plane being

$$-x\bar{y} + y\bar{x} = 0.$$

Putting $Z' = 0$ the first equation gives us

$$(\omega' - \omega) \Sigma mzx = Y'\xi = (\omega' - \omega) \Sigma mx\xi,$$

therefore

$$\Sigma m(z - \xi)x = 0,$$

similarly

$$\Sigma m(z - \xi)y = 0.$$

From these two equations it follows (Art. 38) that the axis of rotation must be a principal axis at a point at a height ξ above the plane of xy , that is, at the point where a plane through the line of action of the impulsive force parallel to the plane of xy cuts the axis of rotation.

The third equation gives us

$$\begin{aligned} (\omega' - \omega) \cdot Mk^2 &= Y'\xi - X'\eta \\ &= (\omega' - \omega) M \cdot (\bar{x}\xi + \bar{y}\eta), \end{aligned}$$

therefore

$$k^2 = \bar{x}\xi + \bar{y}\eta = p\bar{r},$$

where \bar{r} is the distance of the centre of inertia from the axis of rotation $= \sqrt{\bar{x}^2 + \bar{y}^2}$, and p is the perpendicular distance between the axis of rotation and the line of action of the impulsive force. For the equations of this line are

$$\frac{x - \xi}{X'} = \frac{y - \eta}{Y'} = \frac{z - \zeta}{Z'},$$

or

$$\frac{x - \xi}{-\bar{y}} = \frac{y - \eta}{\bar{x}} = \frac{z - \zeta}{0},$$

and the perpendicular distance between this line and the axis of z , which is evidently the perpendicular on this line from the point $(0, 0, \zeta)$ is (Solid Geometry, Art. 28 or 30)

$$\begin{aligned} &= \frac{\bar{x}\xi + \bar{y}\eta}{\sqrt{\bar{x}^2 + \bar{y}^2}}. \end{aligned}$$

These results can be obtained rather more simply by assuming the plane of zx to contain the centre of inertia. This gives us $\bar{y} = 0$ and thence $X' = 0$. Hence the impulse is entirely parallel to the axis of y , that is, is perpendicular to the plane containing the axis of rotation and the centre of inertia. We then easily obtain $k^2 = \bar{x}\xi$, which is equivalent to our last result. It follows that the centre of percussion, if one exist, is at the same distance from the axis of rotation as the centre of oscillation (Art. 56).

EXAMPLES. CHAPTER IV.

1. A body moves about a fixed horizontal axis and is acted on by gravity only, find the time of a small oscillation.

If T_1 and T_2 are the times of a small oscillation about parallel axes which are distant a_1 and a_2 respectively from the centre of gravity; and T the time of a small oscillation for a *simple* pendulum of length $a_1 + a_2$, shew that

$$(a_1 - a_2) T^2 = a_1 T_1^2 - a_2 T_2^2.$$

2. A rigid body is suspended in succession from three parallel axes in the same plane, the distances of which from each other are known. If the times of oscillation be observed, obtain an equation for determining the moment of inertia about the parallel axis through the centre of gravity.

3. A homogeneous solid spheroid, the equation to the surface of which is

$$\frac{x^2}{a^2} + \frac{y^2 + z^2}{b^2} = 1,$$

is suspended from an axis passing through a focus; prove that the centre of oscillation lies on the surface

$$25a^2 (a^2 - b^2) (x^2 + y^2 + z^2)^2 = \{a^2 x^2 + b^2 (x^2 + y^2 + z^2)\}^2.$$

4. A uniform heavy rod OA swings from a hinge at O , and an elastic string is attached to a point C in the rod, the other end of the string being fastened to a point B vertically below O . In the position of equilibrium the string is at its natural length and the coefficient of elasticity is n times the weight of the rod. If the rod be held in a horizontal position and then set free, shew that if ω be the angular velocity when it is vertical

$$\frac{2}{3} a^2 \omega^2 = ga + ng \left(\frac{hc}{h - c} + \sqrt{h^2 + c^2} - h + c \right),$$

where $2a$ = length of rod, $OC = c$, $OB = h$.

Find also the time of a *small* oscillation and shew that it is not affected by the elastic string.

5. A uniform rod of length $2a$ is rotating, in a vertical plane, about its middle point, which is fixed, with an angular velocity $\sqrt{\frac{3\pi g}{a}}$. At the instant the rod is horizontal the ascending end is struck by a ball of equal mass, which was dropped from a height $\frac{\pi a}{3}$; and when it is next horizontal, the same extremity is struck by a second equal ball similarly dropped. The elasticity being perfect, determine the subsequent motion of the rod and balls.

6. A uniform vertical circular plate, of radius a , is capable of revolving about a smooth horizontal axis through its centre; a rough string equal in mass to the plate and in length to its circumference hangs over its rim in equilibrium; if one end be slightly displaced, shew that the velocity of the string when it begins to leave the plate is $\sqrt{\frac{g\pi a}{6}}$.

7. A uniform cylinder can move freely about its axis, which is horizontal. While it is at rest a particle of elasticity e and of $\frac{1}{n^{\text{th}}}$ of its mass and of friction μ falls on it and strikes it with given velocity. Investigate the motion of the cylinder, and shew that the greatest angular velocity will be produced when the radius vector of the particle's point of impact is inclined to the vertical at an angle

$$\tan^{-1} \left\{ \mu \left(\frac{2}{n} \overline{1 + e + 1} \right) \right\}.$$

8. A door in the shape of a uniform rectangle of height h and width b , turns on two hinges in a vertical line at a distance $2c$ apart, and equidistant from the top and bottom. A weight equal to half that of the door is fastened to one end of a string, whose other end is attached to the top corner of the door farthest from the hinges, and which passes over a pulley fixed at the corresponding corner of the doorway. The door is placed open at right angles to the doorway: find the angular velocity with which it comes to.

If the motion be suddenly stopped by a force applied at any point of the door: find the impulsive tension of

the string, and pressures on the hinges, and the condition that the latter may vanish.

9. An elliptic lamina is supported, with its plane vertical and transverse axis horizontal, by two weightless pins passing through its foci. If one of the pins be released, determine the eccentricity of the ellipse in order that the pressure on the other may be initially unaltered.

10. A lamina, whose centre of gravity is G , is revolving about a horizontal axis perpendicular to it and meeting it in C ; supposing it to begin to move from that position in which CG is horizontal, prove that the greatest angle which the direction of the pressure on the axis can make with the vertical is $\cot^{-1} \left(\frac{2}{3} \frac{k^2}{h^2} \tan \theta \right)$, where θ is the corresponding angle which CG makes with the vertical, k is the radius of gyration about the axis through G perpendicular to the lamina, and $CG = h$.

11. A particle is placed on a rough plane lamina which is initially horizontal, and which is moveable about a horizontal line through its centre of gravity. Shew that the particle will begin to slip when the plane has turned through an angle $\tan^{-1} \frac{\mu M a^2}{9 m c^2 + M a^2}$, μ being the coefficient of friction, $2a$ the length of the plane perpendicular to its axis of revolution, c the distance of the particle from that axis, and M and m the masses of the lamina and particle.

12. A ring is constrained to remain in a vertical plane, and to be always in contact with a rough horizontal plane, by passing through a smooth fixed ring at the extremity of a horizontal diameter. A weight equal to $\frac{1}{n}$ th of the weight of the ring is attached to it at the other extremity of the horizontal diameter. Shew that the weight will just reach the horizontal plane if the coefficient of friction be either of the roots of the equation

$$\mu^2 - \mu \frac{\pi}{2} \left\{ \frac{1}{2(n+1)} + (n+1) \right\} + 1 = 0.$$

CHAPTER V.

MOTION OF A BODY WITH ONE POINT FIXED.

62. IF the movement of a body be restricted by having *one* point only fixed, we know by Art. 9, that the motion at any instant can be represented by a rotation round some line passing through the fixed point; or, by the composition of simultaneous angular velocities round the co-ordinate axes. The equations of motion in this case will be, by Art. 23, taking the fixed point as origin

$$\Sigma m \left\{ y \frac{d^2 z}{dt^2} - z \frac{d^2 y}{dt^2} \right\} = L,$$

$$\Sigma m \left\{ z \frac{d^2 x}{dt^2} - x \frac{d^2 z}{dt^2} \right\} = M,$$

$$\Sigma m \left\{ x \frac{d^2 y}{dt^2} - y \frac{d^2 x}{dt^2} \right\} = N,$$

where L , M , N represent the sums of the moments of the impressed forces about the axes of x , y , z respectively.

As in Art. 18, if ω_x , ω_y , ω_z be the angular velocities round the co-ordinate axes, and ω the resultant angular velocity

$$\frac{d^2 x}{dt^2} = p\omega_x - x\omega^2 + z \frac{d\omega_y}{dt} - y \frac{d\omega_z}{dt},$$

$$\frac{d^2 y}{dt^2} = p\omega_y - y\omega^2 + x \frac{d\omega_z}{dt} - z \frac{d\omega_x}{dt};$$

hence

$$x \frac{d^2 y}{dt^2} - y \frac{d^2 x}{dt^2} = (x^2 + y^2) \frac{d\omega_z}{dt} - zx \frac{d\omega_x}{dt} - yz \frac{d\omega_y}{dt} + p(x\omega_y - y\omega_x).$$

Therefore

$$\begin{aligned} \Sigma m \left\{ x \frac{d^2 y}{dt^2} - y \frac{d^2 x}{dt^2} \right\} &= \Sigma m (x^2 + y^2) \frac{d\omega_z}{dt} - \Sigma m zx \frac{d\omega_x}{dt} - \Sigma m yz \frac{d\omega_y}{dt} \\ &+ \Sigma m (x\omega_x + y\omega_y + z\omega_z) (x\omega_y - y\omega_x) = N \dots \dots (1). \end{aligned}$$

Two similar equations can be obtained from the other two above.

63. The equation (1) is ordinarily quite insoluble. In one case however it reduces to a very simple form, that is to say when the three principal moments of inertia of the body at the fixed point are all equal. Any axis is in this case a principal axis, since the momental ellipsoid becomes a sphere (Art. 36). If A be the value of any one of the principal moments, since $\Sigma myz = 0$, $\Sigma mzx = 0$, $\Sigma mxy = 0$, and $\Sigma m (x^2 + y^2) = \Sigma m (y^2 + z^2) = \Sigma m (z^2 + x^2) = A$, the above equation becomes

$$A \frac{d\omega_z}{dt} = N;$$

and the other two become similarly

$$A \frac{d\omega_x}{dt} = L, \quad A \frac{d\omega_y}{dt} = M,$$

and from these equations the values of ω_x , ω_y , ω_z may sometimes be obtained by integration.

64. If a body rotating about a fixed point be subjected to impulsive forces, the equations of Art. 28 enable us to discover the instantaneous alterations in the angular velocities of the body round the axes.

Let ω_x , ω_y , ω_z be the component angular velocities round the axes before, and ω'_x , ω'_y , ω'_z the values of the same quantities after, the impulses. Then with the notation of Art. 28 by means of the formulæ (8) of Art. 7,

$$u = \omega_y z - \omega_z y, \quad v = \omega_z x - \omega_x z, \quad w = \omega_x y - \omega_y x,$$

$$u' = \omega'_y z - \omega'_z y, \quad v' = \omega'_z x - \omega'_x z, \quad w' = \omega'_x y - \omega'_y x.$$

Then equations (1) of Art. 28, give after a little reduction

$$\Sigma m (y^2 + z^2) (\omega_x' - \omega_x) - \Sigma mzx (\omega_z' - \omega_z) - \Sigma mxy (\omega_y' - \omega_y) = \Sigma (Z'y - Y'z),$$

and two similar equations.

If we denote the moments of the impulses round the axes by the symbols L' , M' , N' , and adopt the notation of Art. 34, these equations can be written

$$\left. \begin{aligned} A(\omega_x' - \omega_x) - B'(\omega_z' - \omega_z) - C'(\omega_y' - \omega_y) &= L' \\ B(\omega_y' - \omega_y) - A'(\omega_z' - \omega_z) - C'(\omega_x' - \omega_x) &= M' \\ C(\omega_z' - \omega_z) - A'(\omega_y' - \omega_y) - B'(\omega_x' - \omega_x) &= N' \end{aligned} \right\} \dots\dots\dots(1),$$

which determine $\omega_x' - \omega_x$, $\omega_y' - \omega_y$, $\omega_z' - \omega_z$,

and therefore ω_x' , ω_y' , ω_z' .

65. If a body with one point fixed and initially at rest, be acted on by impulses whose moments round the axes are L' , M' , N' , the angular velocities produced will be obtained from the equations (1) of the last Article, by putting ω_x , ω_y , ω_z each equal to zero.

Now the direction-cosines of the initial instantaneous axis are $\frac{\omega_x'}{\omega'}$, $\frac{\omega_y'}{\omega'}$, $\frac{\omega_z'}{\omega'}$, if ω' be the initial angular velocity

$$= \sqrt{\omega_x'^2 + \omega_y'^2 + \omega_z'^2}.$$

Hence the equation of the plane which is diametral to the initial instantaneous axis with respect to the momental ellipsoid [Art. 35, (3)] is (Solid Geometry, Art. 82),

$$(A\omega_x' - B'\omega_z' - C'\omega_y')x + (B\omega_y' - A'\omega_z' - C'\omega_x')y + (C\omega_z' - A'\omega_y' - B'\omega_x')z = 0.$$

The direction-cosines of the normal to this plane are proportional to

$$\begin{array}{lll} A\omega_x' - B'\omega_z' - C'\omega_y', & B\omega_y' - A'\omega_z' - C'\omega_x', & C\omega_z' - A'\omega_y' - B'\omega_x', \\ \text{or} & L', & M', & N', \end{array}$$

that is to the direction-cosines of the axis of the couple of which L' , M' , N' are components.

Hence if a given set of impulses act on the body, since these can always be replaced by a couple, and a force acting at the fixed point, which latter part has no effect on the rotation; the initial instantaneous axis of rotation is the diameter of the momental ellipsoid which is conjugate to the plane of the couple.

Unless the plane of the couple coincide with a principal plane of the body, the initial axis of rotation will not be perpendicular to the plane of the couple.

The quantities $A\omega_x - B'\omega_z - C'\omega_y$ and the two similar expressions are called the angular momenta of the body about the axes, or the moments of momentum about the axes. (See Art. 97).

66. If at any instant the co-ordinate axes happen to coincide with the principal axes, at that instant $\Sigma myz = 0$, $\Sigma mzx = 0$, $\Sigma mxy = 0$; and if A, B, C be taken as the values of the three principal moments of inertia, we have also

$$\Sigma m(y^2 + z^2) = A, \quad \Sigma m(z^2 + x^2) = B, \quad \Sigma m(x^2 + y^2) = C.$$

The equation (1) of Art. 62 becomes

$$C \frac{d\omega_z}{dt} + \Sigma m(x^2 - y^2) \omega_x \omega_y = N,$$

or
$$C \frac{d\omega_z}{dt} + (B - A) \omega_x \omega_y = N,$$

and the other two equations are similarly simplified.

If however $\omega_x, \omega_y, \omega_z$ represent the angular velocities round axes fixed in space, this simplified form will only be true for the instant considered and will not admit of integration with respect to t . Let us then suppose that the angular velocity is represented by components $\omega_1, \omega_2, \omega_3$ round the principal axes which are fixed in the body and coincide momentarily with the fixed axes. At the instant considered $\omega_x = \omega_1, \omega_y = \omega_2, \omega_z = \omega_3$. We cannot however be sure as to the relation between $\frac{d\omega_1}{dt}, \frac{d\omega_2}{dt}, \frac{d\omega_3}{dt}$ and $\frac{d\omega_x}{dt}, \frac{d\omega_y}{dt}, \frac{d\omega_z}{dt}$, and this relation we must proceed to investigate.

67. Let us then suppose $l_1, m_1, n_1; l_2, m_2, n_2; l_3, m_3, n_3$ to be the direction-cosines of the principal axes of the body with reference to axes fixed in space. We then get (Art. 14),

$$\omega_x = l_1\omega_1 + l_2\omega_2 + l_3\omega_3,$$

$$\omega_y = m_1\omega_1 + m_2\omega_2 + m_3\omega_3,$$

$$\omega_z = n_1\omega_1 + n_2\omega_2 + n_3\omega_3;$$

hence $\frac{d\omega_x}{dt} = l_1 \frac{d\omega_1}{dt} + l_2 \frac{d\omega_2}{dt} + l_3 \frac{d\omega_3}{dt} + \omega_1 \frac{dl_1}{dt} + \omega_2 \frac{dl_2}{dt} + \omega_3 \frac{dl_3}{dt}$.

Turning to Art. 7, we see that the relations between $\omega_1, \omega_2, \omega_3$ and $\omega_x, \omega_y, \omega_z$ are exactly the same in form as those between x', y', z' and ξ, η, ζ .

Hence by a reduction equivalent to that in Art. 7, the last three terms of the value of $\frac{d\omega_x}{dt}$ can be replaced by $\omega_2\omega_y - \omega_y\omega_z$ or zero, since the symbols $\omega_x, \omega_y, \omega_z$ have the same meaning as in Art. 7.

Hence

$$\frac{d\omega_x}{dt} = l_1 \frac{d\omega_1}{dt} + l_2 \frac{d\omega_2}{dt} + l_3 \frac{d\omega_3}{dt}.$$

If then at any instant the principal axes coincide with the fixed axes, at that instant $l_1 = 1, l_2 = 0, l_3 = 0$, and therefore we have

$$\frac{d\omega_x}{dt} = \frac{d\omega_1}{dt}.$$

68. The equations of motion of the body are therefore reduced to

$$A \frac{d\omega_1}{dt} + (C - B) \omega_2 \omega_3 = L,$$

$$B \frac{d\omega_2}{dt} + (A - C) \omega_3 \omega_1 = M,$$

$$C \frac{d\omega_3}{dt} + (B - A) \omega_1 \omega_2 = N,$$

where $\omega_1, \omega_2, \omega_3$ are the resolved angular velocities round the principal axes of the body.

69. In addition to the equations of the last Article which may be supposed to determine $\omega_1, \omega_2, \omega_3$, we still require means for determining the values of $l_1, m_1, n_1; l_2, m_2, n_2; l_3, m_3, n_3$ or in some other way ascertaining the position of the principal axes in space.

It is well known that these nine quantities are connected by six relations. Therefore three independent quantities are required to determine the position of the principal axes. The

Also the angle between the arcs BC and Cz is ψ . Hence resolving the second set in the directions of the first we have

$$\left. \begin{aligned} \omega_1 \cos \psi - \omega_2 \sin \psi &= \frac{d\theta}{dt} \\ \omega_1 \sin \psi + \omega_2 \cos \psi &= \frac{d\phi}{dt} \sin \theta \end{aligned} \right\} \dots\dots\dots(1).$$

From which, or by resolving the first set in the directions of the second, we can obtain the equivalent equations

$$\left. \begin{aligned} \omega_1 &= \frac{d\theta}{dt} \cos \psi + \frac{d\phi}{dt} \sin \theta \sin \psi \\ \omega_2 &= -\frac{d\theta}{dt} \sin \psi + \frac{d\phi}{dt} \sin \theta \cos \psi \end{aligned} \right\} \dots\dots\dots(2).$$

Again, the velocity of A along AB is ω_3 , but its velocity along AB relative to x_1 , is $\frac{d\psi}{dt}$, while the velocity of x_1 along AB is $\frac{d\phi}{dt} \cos \theta$.

Hence we must have,

$$\omega_3 = \frac{d\psi}{dt} + \frac{d\phi}{dt} \cos \theta \dots\dots\dots(3).$$

The equations (1) or (2) with (3) theoretically determine θ , ϕ , ψ if ω_1 , ω_2 , ω_3 are known, and, coupled with the equations of Art. 68, determine the motion completely.

71. In the case of a body acted on, either by no forces, or by forces which have no moment round the fixed point, the equations of Art. 68 admit of integration. They become

$$A \frac{d\omega_1}{dt} + (C - B) \omega_2 \omega_3 = 0 \dots\dots\dots(1),$$

$$B \frac{d\omega_2}{dt} + (A - C) \omega_3 \omega_1 = 0 \dots\dots\dots(2),$$

$$C \frac{d\omega_3}{dt} + (B - A) \omega_1 \omega_2 = 0 \dots\dots\dots(3).$$

Multiplying these equations by ω_1 , ω_2 , ω_3 respectively, and adding, we have,

$$A\omega_1 \frac{d\omega_1}{dt} + B\omega_2 \frac{d\omega_2}{dt} + C\omega_3 \frac{d\omega_3}{dt} = 0,$$

whence $A\omega_1^2 + B\omega_2^2 + C\omega_3^2 = \text{constant} = h^2$ suppose ... (4).

Again multiplying the three original equations by $A\omega_1$, $B\omega_2$, $C\omega_3$ respectively and adding, we obtain

$$A^2\omega_1 \frac{d\omega_1}{dt} + B^2\omega_2 \frac{d\omega_2}{dt} + C^2\omega_3 \frac{d\omega_3}{dt} = 0,$$

whence $A^2\omega_1^2 + B^2\omega_2^2 + C^2\omega_3^2 = \text{constant} = k^4$ suppose... (5).

Let us further assume ω to represent the resultant angular velocity, so that

$$\omega_1^2 + \omega_2^2 + \omega_3^2 = \omega^2 \dots \dots \dots (6).$$

From these three equations ω_1^2 , ω_2^2 , ω_3^2 can be expressed in terms of ω , and since

$$\omega \frac{d\omega}{dt} = \omega_1 \frac{d\omega_1}{dt} + \omega_2 \frac{d\omega_2}{dt} + \omega_3 \frac{d\omega_3}{dt} \dots \dots \dots (7),$$

we can then by means of (1), (2) and (3) express $\frac{d\omega}{dt}$, and

consequently $\frac{dt}{d\omega}$, in terms of ω . The value of t is thus made to depend on the determination of an elliptic integral.

72. The equations of the instantaneous axis referred to the principal axes are

$$\frac{x}{\omega_1} = \frac{y}{\omega_2} = \frac{z}{\omega_3} \dots \dots \dots (1).$$

The co-ordinates of the point where (1) meets the momental ellipsoid, whose equation is

$$Ax^2 + By^2 + Cz^2 = \epsilon^4 \dots \dots \dots (2),$$

are obtained by taking (1) and (2) as simultaneous. We thus get

$$\frac{x}{\omega_1} = \frac{y}{\omega_2} = \frac{z}{\omega_3} = \sqrt{\frac{Ax^2 + By^2 + Cz^2}{A\omega_1^2 + B\omega_2^2 + C\omega_3^2}} = \frac{\epsilon^2}{h},$$

and also each of the fractions

$$= \sqrt{\frac{x^2 + y^2 + z^2}{\omega_1^2 + \omega_2^2 + \omega_3^2}} = \frac{r}{\omega}$$

if r be the distance of the point from the origin.

The equation of the tangent plane to (2) at this point is (Solid Geometry, Art. 101),

$$\frac{A\omega_1\epsilon^2}{h}x + \frac{B\omega_2\epsilon^2}{h}y + \frac{C\omega_3\epsilon^2}{h}z = \epsilon^4,$$

or
$$A\omega_1x + B\omega_2y + C\omega_3z = \epsilon^2h \dots\dots\dots(3).$$

The length of the perpendicular from the origin on this plane is therefore

$$= \frac{h\epsilon^2}{\sqrt{A^2\omega_1^2 + B^2\omega_2^2 + C^2\omega_3^2}} = \frac{h\epsilon^2}{k^2},$$

and is consequently invariable.

73. The direction-cosines of the perpendicular on the tangent plane (3) referred to the principal axes are

$$\frac{A\omega_1}{k^2}, \frac{B\omega_2}{k^2}, \frac{C\omega_3}{k^2}.$$

We shall now shew that this line is not only invariable in length but also in absolute direction.

Let l_1, m_1, n_1 be the direction-cosines of OA with respect to axes Ox, Oy, Oz fixed in space; l_2, m_2, n_2 ; l_3, m_3, n_3 those of OB and OC .

The cosine of the angle between Ox and the line we are considering is $\frac{A\omega_1l_1 + B\omega_2l_2 + C\omega_3l_3}{k^2} = \lambda$ say.

From this we obtain by differentiation

$$\begin{aligned} k^2 \frac{d\lambda}{dt} &= Al_1 \frac{d\omega_1}{dt} + Bl_2 \frac{d\omega_2}{dt} + Cl_3 \frac{d\omega_3}{dt} + A\omega_1 \frac{dl_1}{dt} + B\omega_2 \frac{dl_2}{dt} + C\omega_3 \frac{dl_3}{dt} \\ &= (B-C)l_1\omega_2\omega_3 + (C-A)l_2\omega_3\omega_1 + (A-B)l_3\omega_1\omega_2 \\ &\quad + A\omega_1 \frac{dl_1}{dt} + B\omega_2 \frac{dl_2}{dt} + C\omega_3 \frac{dl_3}{dt} \end{aligned}$$

$$\begin{aligned}
&= A\omega_1 \left(\frac{dl_1}{dt} - l_2\omega_3 + l_3\omega_2 \right) + B\omega_2 \left(\frac{dl_2}{dt} - l_3\omega_1 + l_1\omega_3 \right) \\
&\quad + C\omega_3 \left(\frac{dl_3}{dt} - l_1\omega_2 + l_2\omega_1 \right).
\end{aligned}$$

But by Art. 7 if $\omega_x, \omega_y, \omega_z$ are the angular velocities round Ox, Oy, Oz we have

$$l_1 \frac{dl_1}{dt} + l_2 \frac{dl_2}{dt} + l_3 \frac{dl_3}{dt} = 0,$$

$$m_1 \frac{dl_1}{dt} + m_2 \frac{dl_2}{dt} + m_3 \frac{dl_3}{dt} = -\omega_z,$$

$$n_1 \frac{dl_1}{dt} + n_2 \frac{dl_2}{dt} + n_3 \frac{dl_3}{dt} = \omega_y,$$

and multiplying these equations by l_1, m_1, n_1 respectively and adding we obtain

$$\frac{dl_1}{dt} = n_1\omega_y - m_1\omega_z$$

$$= n_1(m_1\omega_1 + m_2\omega_2 + m_3\omega_3) - m_1(n_1\omega_1 + n_2\omega_2 + n_3\omega_3) \text{ (Art. 67),}$$

$$= \omega_3(m_3n_1 - m_1n_3) - \omega_2(m_1n_2 - m_2n_1),$$

$$= l_2\omega_3 - l_3\omega_2 \text{ (Solid Geometry, Chap. IV. Ex. 4).}$$

$$\text{Similarly } \frac{dl_2}{dt} = l_3\omega_1 - l_1\omega_3,$$

$$\frac{dl_3}{dt} = l_1\omega_2 - l_2\omega_1.$$

Hence $\frac{d\lambda}{dt} = 0$; that is the perpendicular to the tangent

plane of the momental ellipsoid at the extremity of the instantaneous axis makes a constant angle with any one, and consequently with all, of the axes fixed in space. Hence this line is itself fixed in direction. It is therefore called the invariable line, and the plane through the fixed point perpendicular to it is called the invariable plane. Another proof of this result will be given later on as a particular case of a more general proposition (Art. 101).

74. If the invariable line be taken as axis of z in the figure of Art. 70 the equations which give the motion can be considerably simplified.

For from spherical trigonometry or from the transformations effected in Art. 45 of the author's Solid Geometry the direction-cosines of Oz with reference to OA , OB , OC are

$$\sin \psi \sin \theta, \cos \psi \sin \theta, \cos \theta.$$

Hence

$$\frac{A\omega_1}{k^2} = \sin \psi \sin \theta, \frac{B\omega_2}{k^2} = \cos \psi \sin \theta, \frac{C\omega_3}{k^2} = \cos \theta \dots (1).$$

Equations (1) of Art. 70 therefore give

$$\frac{d\theta}{dt} = \left(\frac{k^2}{A} - \frac{k^2}{B} \right) \sin \psi \cos \psi \sin \theta \dots \dots \dots (2),$$

$$\frac{d\phi}{dt} = k^2 \left(\frac{\cos^2 \psi}{B} + \frac{\sin^2 \psi}{A} \right) \dots \dots \dots (3),$$

while equation (3) gives

$$\frac{k^2}{C} \cos \theta = \frac{d\psi}{dt} + k^2 \left(\frac{\cos^2 \psi}{B} + \frac{\sin^2 \psi}{A} \right) \cos \theta,$$

$$\frac{d\psi}{dt} = k^2 \cos \theta \left\{ \left(\frac{1}{C} - \frac{1}{A} \right) \sin^2 \psi + \left(\frac{1}{C} - \frac{1}{B} \right) \cos^2 \psi \right\} \dots (4).$$

Equations (2), (3), (4) if integrable would give the values of θ , ψ , ϕ in terms of t , and so determine the position of the body at any time.

It is perhaps worth noticing that the direction-cosines of the three lines Ox , Oy , Oz with respect to OA , OB , OC are

$$\left. \begin{array}{l} \sin \phi \cos \psi - \sin \phi \sin \psi \cos \theta, -\cos \phi \sin \psi - \sin \phi \cos \psi \cos \theta, \sin \phi \sin \theta \\ \sin \phi \cos \psi + \cos \phi \sin \psi \cos \theta, -\sin \phi \sin \psi + \cos \phi \cos \psi \cos \theta, \cos \phi \sin \theta \\ \sin \psi \sin \theta, \cos \psi \sin \theta, \cos \theta \end{array} \right\} (5),$$

and that the same nine quantities taken in vertical rows are the direction-cosines of OA , OB , OC with respect to Ox , Oy , Oz . Also that the direction-cosines of OB can be

deduced from those of OA by writing $\psi + \frac{\pi}{2}$ for ψ while those of Oy can be deduced from those of Ox by writing

$$\phi - \frac{\pi}{2} \text{ for } \phi.$$

75. The motion of the body can thus be represented by imagining the momental ellipsoid at the fixed point to roll on a plane parallel to the invariable plane at a distance $\frac{h\epsilon^2}{k^2}$ from the origin. The ellipsoid *rolls* on the plane because the point of contact at each instant, being the extremity of the instantaneous axis, has no velocity in space.

The instantaneous axis describes two cones, one absolutely in space whose base is a curve traced out on the fixed tangent plane by the successive positions of the point of contact; and the other relatively to the momental ellipsoid, whose base is the curve traced out by the same point on the surface of the ellipsoid. The motion of the body may be also represented by imagining one of these cones to roll on the other.

76. The locus of the extremities of the instantaneous axis on the ellipsoid is easily obtained. It is simply the locus of points at which the perpendicular on the tangent plane is equal to $\frac{h\epsilon^2}{k^2}$, and the condition for this is that

$$A^2x^2 + B^2y^2 + C^2z^2 = \frac{k^4\epsilon^4}{h^2},$$

which, combined with the equation

$$Ax^2 + By^2 + Cz^2 = \epsilon^4,$$

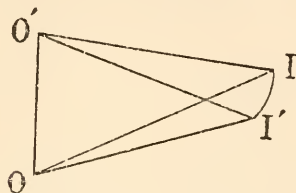
gives the curve required.

This curve is called the *polhode*.

77. The equation of the curve, called the *herpolhode*, traced out on the fixed tangent plane can be obtained.

If ds' and ds be corresponding elements of the arc of this curve and the polhode, it is evident from the method of description that $ds' = ds$.

Let O be the fixed point, OI any position of the instantaneous axis, $O'I$ the next position. We may suppose II' to be an arc of the polhode or herpolhode. Let OO' be the



invariable line, and O' the point where it meets the tangent plane to the ellipsoid at I . Then if $O'I = r'$, $OI = r$, and θ' be the angle which $O'I$ makes with some fixed line in the plane $IO'I$,

$$\delta s'^2 = \delta s^2 = II'^2 = \delta r'^2 + r'^2 (\delta \theta')^2.$$

But if x, y, z be the co-ordinates of I referred to the principal axes, $(\delta s)^2 = (\delta x)^2 + (\delta y)^2 + (\delta z)^2$

$$= \frac{\epsilon^4}{h^2} \{(\delta \omega_1)^2 + (\delta \omega_2)^2 + (\delta \omega_3)^2\}, \text{ by Art. 72,}$$

therefore proceeding to the limit

$$\left(\frac{dr'}{dt}\right)^2 + r'^2 \left(\frac{d\theta'}{dt}\right)^2 = \frac{\epsilon^4}{h^2} \left\{ \left(\frac{d\omega_1}{dt}\right)^2 + \left(\frac{d\omega_2}{dt}\right)^2 + \left(\frac{d\omega_3}{dt}\right)^2 \right\} \dots (1).$$

Also since $OO' = h \frac{\epsilon^2}{k^2}$, Art. 72,

$$r^2 = r'^2 + \frac{h^2 \epsilon^4}{k^4},$$

or
$$r'^2 = r^2 - \frac{h^2 \epsilon^4}{k^4} = \frac{\omega^2 \epsilon^4}{h^2} - \frac{h^2 \epsilon^4}{k^4},$$

whence

$$r' \frac{dr'}{dt} = \frac{\omega \epsilon^4}{h^2} \frac{d\omega}{dt} \dots \dots \dots (2).$$

By Art. 71 we can express the right hand member of (1) in terms of ω . Hence from (1) and (2) we can express $\frac{dr'}{dt}$ and $\frac{d\theta'}{dt}$, and therefore $\frac{dr'}{d\omega}$ and $\frac{d\theta'}{d\omega}$, in terms of ω . Whence by integrating and eliminating ω , r' can be obtained in terms of θ' .

We may notice that the quantity denoted by h^2 is the *vis viva* of the body (Arts. 91, 96); while that denoted by k^2 is the moment of momentum, or the angular momentum of the body, round the invariable line (Arts. 88, 100, 101).

EXAMPLES. CHAPTER V.

1. An ellipsoid rotating, with its centre fixed, about one of its principal axes (that of x) receives a normal blow at a point (h, k, l) . If the initial axis of rotation after the blow lie in the principal plane of yz , its equation is

$$c^2 (a^2 + c^2) (a^2 - b^2) ky + b^2 (a^2 + b^2) (a^2 - c^2) lz = 0.$$

2. The angular velocities of a body acted on by couples L, M, N round the principal axes, about which the moments of inertia are A, B, C , are $\omega_1, \omega_2, \omega_3$; shew that the angular velocities, in the body, of the instantaneous axis round the principal axes are

$$\frac{1}{\omega_2^2 + \omega_3^2} \left\{ \frac{\omega_2 N}{C} - \frac{\omega_3 M}{B} + \omega_1 \left(\frac{C - A}{B} \omega_3^2 - \frac{A - B}{C} \omega_2^2 \right) \right\},$$

and similar expressions.

3. If A, B, C are the moments of inertia, $-F, -G, -H$ the products of inertia, for rectangular axes fixed in space, of a rigid body rotating about a fixed origin, $\omega_1, \omega_2, \omega_3$ the component angular velocities, and K_1, K_2, K_3 the component angular momenta about the same axes at the time t ; if $L = BC - F^2, P = GH - AF$, prove that

$$\frac{dA}{dt} = 2 (G\omega_2 - H\omega_3), \quad \frac{dF}{dt} = (B - C) \omega_1 - H\omega_2 + G\omega_3,$$

$$\frac{dL}{dt} = 2 (HK_3 - GK_2), \quad \frac{dP}{dt} = (C - B) K_1 + HK_2 - GK_3.$$

4. If two of the principal moments of inertia be equal, and the body begin to rotate about an axis perpendicular to that of unequal moment, under the action of a couple varying as the cosecant of the angle which the instantaneous axis makes with the axis of unequal moment, and in a plane perpendicular to that axis, determine the position of the instantaneous axis in the body in terms of the time.

5. If, with the usual notation, $M=0$, $N=0$, and the body initially rotate about the principal axis to which A belongs, it will continue to do so if A be the greatest or least moment, but if it be the mean moment, the axis of rotation will always be in a plane through this axis whose equation is

$$y\sqrt{B(A-B)} = z\sqrt{C(C-A)}.$$

6. A lamina of any form, rotating with an angular velocity ω about an axis through its centre of gravity perpendicular to its plane, has an angular velocity,

$$\left(\frac{A+B}{A-B}\right)^{\frac{1}{2}}\omega,$$

impressed upon it about its principal axis of least moment, A and B being its moments of inertia about the principal axes of mean and least moment; shew that its angular velocities about the principal axes at any time t , are

$$\frac{2\omega}{\epsilon^{\omega t} + \epsilon^{-\omega t}}, \quad -\left(\frac{A+B}{A-B}\right)^{\frac{1}{2}}\omega \cdot \frac{\epsilon^{\omega t} - \epsilon^{-\omega t}}{\epsilon^{\omega t} + \epsilon^{-\omega t}}, \quad \left(\frac{A+B}{A-B}\right)^{\frac{1}{2}} \frac{2\omega}{\epsilon^{\omega t} + \epsilon^{-\omega t}},$$

and that it will ultimately revolve about its axis of mean moment.

7. A rigid body, acted on by no force, moves in such a manner that the square of its angular momentum is equal to its *vis viva* multiplied by its moment of inertia about its mean axis; prove that the plane through the invariable line and the mean axis rotates uniformly in space, and that, if θ be the inclination of the mean axis to the invariable line at the time t ,

$$\log \tan \frac{\theta}{2} = \log \tan \frac{\alpha}{2} - \frac{(A-B)^{\frac{1}{2}}(B-C)^{\frac{1}{2}}}{B(AC)^{\frac{1}{2}}} Gt,$$

where A, B, C are the principal moments of inertia of the body, G its angular momentum, and α the initial value of θ .

8. If $k^4 = Bh^2$, prove that the polhode becomes a pair of plane curves formed by the intersection of the planes $x\sqrt{A(A-B)} = \pm z\sqrt{C(B-C)}$ with the momental ellipsoid, B being the mean of the quantities A, B, C .

CHAPTER VI.

ON MOTION OF A FREE BODY IN SPACE. THE MOTION OF A SPHERE.

78. THE principles and results of the last chapter can all be utilised for the investigation of the motion of a body no point of which is fixed, by means of the two principles of Art. 24. All the results of Arts. 71—77, apply to the case of a rigid body moving freely and acted on only by forces whose resultant has no moment round the centre of inertia. The only change is one of interpretation, that motions which were formerly absolute are now to be considered merely as relative to the centre of inertia.

79. There is one case of sufficient importance and frequency of occurrence to deserve separate mention, that of a body in the shape of a plane lamina moving in its own plane.

In this case a line through the centre of inertia perpendicular to the plane is a principal axis (Art. 40), and if ω be the angular velocity of the body round this axis, or as it may be called, the angular velocity of the body in its own plane round the centre of inertia, we have by equation (1) of Art. 62, since $\omega_x = 0, \omega_y = 0, \omega_z = \omega$

$$Mk^2 \frac{d\omega}{dt} = \Sigma m (Yx - Xy) \dots \dots \dots (1),$$

if Mk^2 be the moment of inertia of the body about the line through the centre of inertia perpendicular to its plane.

Also if \bar{x}, \bar{y} be the co-ordinates of the centre of inertia referred to axes fixed in the plane of motion; by equations (4) of Art. 25,



$$\left. \begin{aligned} M \frac{d^2 \bar{x}}{dt^2} &= \Sigma m X \\ M \frac{d^2 \bar{y}}{dt^2} &= \Sigma m Y \end{aligned} \right\} \dots \dots \dots (2).$$

These results apply also to the motion of a body not a plane lamina, when the motion of every particle is parallel to the plane of xy .

80. If a plane lamina be acted on by impulsive forces in its own plane; the equation giving the motion round the centre of inertia after the impulse is by Art. 64,

$$Mk^2 (\omega' - \omega) = \Sigma (Y'x - X'y),$$

where ω, ω' are the angular velocities before and after; X', Y' are the resolved impulses at the point (x, y) .

For the motion of the centre of inertia, we have by Art. 28,

$$M(\bar{u}' - \bar{u}) = \Sigma X', \quad M(\bar{v}' - \bar{v}) = \Sigma Y'.$$

The resolved parts of the velocity of any point of the lamina parallel to the axes of x and y after the impulses are then $\bar{u}' - y\omega'$ and $\bar{v}' + x\omega'$.

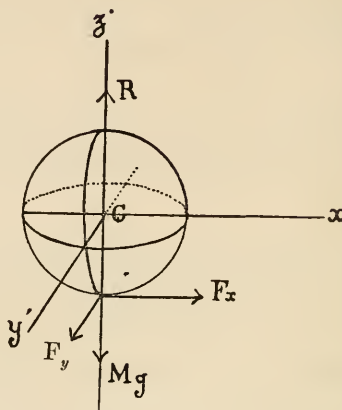
If x, y be such that these both vanish the point (x, y) is instantaneously at rest and is called the centre of instantaneous or spontaneous rotation.

81. Let us suppose a sphere rolling on a horizontal plane under the action of no forces but gravity.

We may take any point in the horizontal plane as origin and any two lines at right angles to each other in that plane as axes of x and y , the axis of z being consequently vertical.

We will assume a to be the radius of the sphere and x, y, a as the co-ordinates of the centre of the sphere. Also let $\omega_x, \omega_y, \omega_z$ be the angular velocities of the sphere at any instant round axes through its centre parallel to the co-ordinate axes. Let R be the reaction of the plane and F'_x, F'_y the resolved parts of the friction parallel to the co-ordinate axes. Let M be the mass of the sphere and Mk^2 its moment of inertia round any diameter.

The principles of Art. 24, give for the motion of the centre of the sphere



$$\left. \begin{aligned} M \frac{d^2 x}{dt^2} &= F_x \\ M \frac{d^2 y}{dt^2} &= F_y \\ 0 &= R - Mg \end{aligned} \right\} \dots\dots\dots(1).$$

For the motion round the centre we have by Art. 63,

$$\left. \begin{aligned} Mk^2 \frac{d\omega_x}{dt} &= F_y a \\ Mk^2 \frac{d\omega_y}{dt} &= -F_x a \\ Mk^2 \frac{d\omega_z}{dt} &= 0 \end{aligned} \right\} \dots\dots\dots(2).$$

These six equations are all the dynamical ones that can be obtained. As there are eight unknowns

$$\omega_x, \omega_y, \omega_z, x, y, F_x, F_y, R,$$

we require two more equations. These must be geometrical and be obtained from the nature of the rolling.

Let us first suppose that the friction of the plane is sufficient to prevent all sliding. In this case the instantaneous velocity of the point of the sphere in contact with the plane must vanish. By Art. 8, the velocities of this point parallel to Ox, Oy , are $\frac{dx}{dt} - a\omega_y$ and $\frac{dy}{dt} + a\omega_x$.

Hence we have

$$\left. \begin{aligned} \frac{dx}{dt} - a\omega_y &= 0 \\ \frac{dy}{dt} + a\omega_x &= 0 \end{aligned} \right\} \dots\dots\dots(3),$$

as the two additional conditions required.

Secondly, it may happen that the friction of the plane is not sufficient to insure perfect rolling. In this case the maximum amount of friction will be called into play in order to prevent the motion of the point of the sphere which is in contact with the plane. Hence, as a first condition, if μ be the coefficient of friction between the sphere and the plane,

$$F_x^2 + F_y^2 = \mu^2 R^2 \dots\dots\dots(4).$$

A second property of friction is, that it always acts in a direction opposite to that in which motion is taking, or tends to take place. This gives as a second condition

$$\frac{F_y}{F_x} = \frac{\frac{dy}{dt} + a\omega_x}{\frac{dx}{dt} - a\omega_y} \dots\dots\dots(5).$$

82. From the first equation of (1) combined with the second of (2), we get

$$\left. \begin{aligned} a \frac{d^2x}{dt^2} + k^2 \frac{d\omega_y}{dt} &= 0 \\ a \frac{d^2y}{dt^2} - k^2 \frac{d\omega_x}{dt} &= 0 \end{aligned} \right\} \dots\dots\dots(6).$$

Similarly

And if we assume the condition of perfect rolling, and substitute for ω_x , ω_y their values from (3), these equations give

$$\frac{d^2x}{dt^2} = 0, \quad \frac{d^2y}{dt^2} = 0.$$

Whence $F_x = 0, \quad F_y = 0.$

If therefore the ball be at any time rolling without sliding, it will continue to do so throughout the motion and the linear and angular velocities will remain uniform.

83. If when the ball is started the impulsive frictional action be not enough to ensure initial perfect rolling, the solution will be somewhat different.

We have from equations (1) and (2)

$$\frac{F_y}{F_x} = \frac{\frac{d^2y}{dt^2}}{\frac{d^2x}{dt^2}} = \frac{\frac{d\omega_x}{dt}}{-\frac{d\omega_y}{dt}} = \frac{\frac{d^2y}{dt^2} + a \frac{d\omega_x}{dt}}{\frac{d^2x}{dt^2} - a \frac{d\omega_y}{dt}},$$

and by (5)

$$\frac{F_y}{F_x} = \frac{\frac{dy}{dt} + a\omega_x}{\frac{dx}{dt} - a\omega_y}.$$

Therefore

$$\frac{\frac{d^2y}{dt^2} + a \frac{d\omega_x}{dt}}{\frac{dy}{dt} + a\omega_x} = \frac{\frac{d^2x}{dt^2} - a \frac{d\omega_y}{dt}}{\frac{dx}{dt} - a\omega_y}.$$

Whence, integrating,

$$\log \left(\frac{dy}{dt} + a\omega_x \right) = \log \left(\frac{dx}{dt} - a\omega_y \right) + \text{constant};$$

therefore

$$\frac{dy}{dt} + a\omega_x = C \left(\frac{dx}{dt} - a\omega_y \right);$$

therefore

$$F_y = C \cdot F_x.$$

And therefore from (4)

$$F_x^2 (1 + C^2) = \mu^2 R^2 = \mu^2 M^2 g^2;$$

therefore

$$F_x = \frac{\mu M g}{\sqrt{1 + C^2}}, \quad F_y = \frac{C \mu M g}{\sqrt{1 + C^2}}.$$

Hence the motion of the centre of the sphere is the same as that of a particle acted on by a constant force in a constant direction, and its path is therefore a parabola, with a straight line as a particular case.

84. If the plane on which the sphere is rolling be inclined to the horizon at an angle α , and we take the line of

greatest slope in this plane as axis of x , the axis of y being consequently horizontal, the equations (1) of Art. 81 are replaced by

$$\left. \begin{aligned} M \frac{d^2x}{dt^2} &= F_x - Mg \sin \alpha \\ M \frac{d^2y}{dt^2} &= F_y \\ 0 &= R - Mg \cos \alpha \end{aligned} \right\} \dots\dots\dots (6).$$

The equations (2) require no alteration.

Eliminating F_x between the first equation of (6) and the second of (2), we obtain

$$k^2 \frac{d\omega_y}{dt} + a \frac{d^2x}{dt^2} = -ag \sin \alpha.$$

Similarly $k^2 \frac{d\omega_x}{dt} - a \frac{d^2y}{dt^2} = 0.$

If the conditions of perfect rolling be satisfied, substituting for ω_x and ω_y we get

$$\begin{aligned} \frac{d^2x}{dt^2} &= -\frac{a^2g \sin \alpha}{a^2 + k^2}, \\ \frac{d^2y}{dt^2} &= 0. \end{aligned}$$

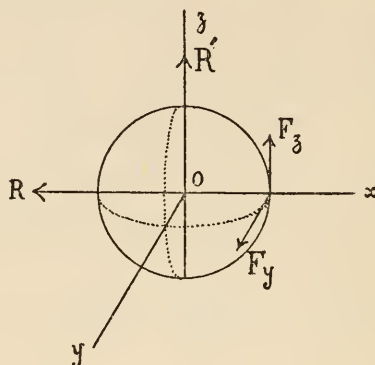
Hence the path of the centre of the sphere is a parabola, since these equations are the same in form as those which give the motion of a particle acted on by a constant force in a constant direction.

85. The preceding Articles furnish sufficient exemplifications of the motion of a sphere acted on by finite forces. As an example of the treatment of the question when impulsive forces come into play we will suppose a sphere moving in any manner on a smooth horizontal plane to strike a vertical rough plane.

We will take the normal to this latter plane as axis of x , and the axis of z vertical.

Let u, v be the velocities of the centre of the sphere parallel to the axes of x and y before the impact; u', v' the

values of the same quantities after impact. Let $\omega_x, \omega_y, \omega_z$ be the angular velocities of the sphere round axes through its centre parallel to the co-ordinate axes before the impact, $\omega'_x, \omega'_y, \omega'_z$ the values of the same quantities after.



Let R, R' be the impulses in the direction of the normal on the vertical and horizontal planes respectively, F_y, F_z the frictional impulses on the former plane parallel to the axes of y and z respectively. Then, assuming that the ball does not jump from the horizontal plane, by equations (2) of Art. 28

$$M(u' - u) = -R; \quad M(v' - v) = F_y; \quad 0 = R' + F_z.$$

And by equations (1) of Art. 64

$$Mk^2(\omega'_x - \omega_x) = 0, \quad Mk^2(\omega'_y - \omega_y) = -aF_z, \quad Mk^2(\omega'_z - \omega_z) = aF_y.$$

If we suppose the vertical plane sufficiently rough to prevent any sliding, we have, by the formulæ of Art. 8,

$$u' = 0, \quad v' + a\omega'_z = 0, \quad -a\omega'_y = 0.$$

These nine equations enable us to determine

$$u', v', \omega'_x, \omega'_y, \omega'_z, R, R', F_y, F_z.$$

We easily obtain

$$v' - v = \frac{k^2}{a}(\omega'_z - \omega_z) = -\frac{k^2}{a^2}v' - \frac{k^2}{a}\omega_z;$$

therefore
$$v' = \frac{a(av - k^2\omega_z)}{a^2 + k^2}; \quad \omega'_z = \frac{k^2\omega_z - av}{a^2 + k^2},$$

and the other quantities can be easily found.

86. We have supposed the sphere inelastic. If this be not the case we know that a new normal reaction $= eR$ comes into play. This will not affect either the dynamical or geometrical conditions which determine v' or ω'_z . It will alter the velocity of the centre of inertia parallel to Ox . If u'' be the final value of this velocity we shall have

$$M(u'' - u') = eR.$$

And, since $u' = 0$, this gives $Mu'' = eR = -eMu$;
therefore $u'' = -eu$.

If ϕ, ϕ' be the angles which the line of motion of the centre of the sphere makes with the normal to the vertical wall before and after impact,

$$\tan \phi = \frac{v}{u}, \quad \tan \phi' = \frac{v'}{-u''}.$$

Hence
$$\tan \phi' = \frac{a(av - k^2\omega_z)}{eu(a^2 + k^2)} = \frac{a^2 \left(1 - \frac{k^2\omega_z}{av}\right)}{e(a^2 + k^2)} \tan \phi,$$

or, since $k^2 = \frac{2a^2}{5}$ (Art. 47),

$$\tan \phi' = \frac{5 - \frac{2a\omega_z}{v}}{7e} \tan \phi.$$

The ratio of $\tan \phi'$ to $\tan \phi$ may thus be made to assume any desired value by properly adjusting the value of $\frac{a\omega_z}{v}$.

87. If there be a number of bodies connected together in any way and mutually acting on one another, the principles laid down will enable the student at least to obtain the equations of motion. These must be written down for each body separately: three for the motion of the centre of inertia of that body and three for the motion round it. If there be n bodies there will consequently be $6n$ dynamical equations. In these equations will occur the unknown forces of constraint between the different bodies. Each such force will however bring in a corresponding limitation of the geometric conditions of the question, and from this limitation we shall be able to find a geometric condition between some of the $6n$ quantities

theoretically necessary to determine the positions of the n bodies. There will thus always be as many equations as unknown quantities, and the problem is therefore reduced to a question of the powers of analysis to solve these equations.

There are certain general principles which will in most cases enable us to obtain one or more integrals of the equations of motion. These form the subject of the next chapter.

EXAMPLES. CHAPTER VI.

1. Shew that a lamina rotating about an axis in its plane can generally be reduced to rest by a single blow.

2. A plane lamina moving, either about a fixed axis or instantaneously about a principal axis, impinges on a free inelastic particle in the line through its centre of gravity perpendicular to the axis of rotation at the time of impact. If the velocity of the particle after impact be the maximum velocity, prove that the angular velocity of the lamina will be diminished in the ratio of 2 : 1.

3. A uniform rod of given length and mass is hinged at one end to a fixed point and has a string fastened to its other end passing over a pulley in the same horizontal line with the fixed point and at a given distance from it greater than the length of the rod. At the other end of the string hangs a given weight. Initially the rod is horizontal. Find how far the weight will ascend.

4. An inelastic rod of length $2a$ falls inclined at an angle θ with the vertical and strikes a smooth horizontal plane. Shew that it immediately acquires an angular velocity

$$\frac{V}{a} \cdot \frac{3 \sin \theta}{(1 + 3 \sin^2 \theta)}, \quad V \text{ being its previous velocity.}$$

5. A uniformly revolving rod, the centre of gravity of which is initially at rest, moves in a plane under the action of a constant force in the direction of its length: prove that, at the end of any time from the beginning of the motion, the square of the radius of curvature of the path of the rod's centre of gravity varies as the versed sine of the angle through which the rod has revolved.

6. A wheel, of mass M and radius c , rotates in a vertical plane about its centre, which is fixed. A heavy uniform rod, of mass M' and length $2a$, has one extremity fastened to a point in the circumference of the wheel and the other moves freely in a vertical smooth groove passing through the centre of the wheel. Determine the motion.

7. A smooth semi-circular disc rests with its plane vertical on a smooth horizontal table, and on it rest two equal uniform rods, each of which passes through two fixed rings in a vertical line. If the disc be slightly displaced, and if, in the ensuing motion, one rod leave the disc when the other is at the highest point, prove that

$$\frac{M}{m} = \frac{2(2 \sin \alpha - 1 - \sin \beta) - \sin \beta \cos \beta}{\sin^3 \beta},$$

M, m being the masses of the disc, and either of the rods respectively, α the angle which the radius to either point of contact, in the position of equilibrium, makes with the horizon, and β being equal to $\cos^{-1}(2 \cos \alpha)$.

8. A circular disc falling vertically impinges with velocity v on a rough obstacle at a point at an angular distance of 45° from the lowest point of the disc. If the coefficients of normal and tangential elasticity be each unity, prove that the latus rectum of the subsequent path of the centre of the disc is $\frac{8}{9} \frac{v^2}{g}$.

9. Explain the method of treating questions involving the rolling and slipping of one rough cylinder on another.

A uniform circular ring moves on a rough curve under the action of no forces, the curvature of the curve being everywhere less than that of the ring. If the ring be projected without rotation from a point A of the curve and begin to roll at a point B , the angle between the normals at A and B is $\log 2 \div \mu$, where μ is the coefficient of friction.

10. A sphere is partly rolling and partly sliding on a rough horizontal plane. Shew that the angle the direction

of friction makes with the axis of x is $\tan^{-1} \frac{u + a\Omega_1}{v - a\Omega_2}$, u and v being the initial velocities, Ω_1 and Ω_2 the initial angular velocities.

11. A circular disc, capable of motion about a vertical axis through its centre perpendicular to its plane, is set in motion with angular velocity Ω , and at a point very near the centre of the disc is placed a rough uniform sphere; shew that when the sphere leaves the disc, the angular velocity of the disc is

$$\frac{7m\Omega}{7m + 4M},$$

M, m being the masses of the sphere and disc.

12. A ball spinning about a vertical axis moves on a smooth table and impinges directly on a perfectly rough vertical cushion; shew that the *vis viva* of the ball is diminished in the ratio $10e^2 + 14\tan^2\theta : 10 + 49\tan^2\theta$, where e is the elasticity of the ball and θ the angle of reflexion.

13. A billiard ball, of radius a , rolling in a straight line with velocity V , and rotating with angular velocity w about a vertical diameter, strikes a cushion at an angle α . The coefficients of normal and frictional elasticity between the ball and cushion are the same, and the friction is sufficient to prevent sliding, the table being supposed smooth. Shew that whatever be the elasticity, the ball after impact will return along its former path, if

$$2aw = 5V \cos \alpha.$$

14. Shew that if the table be rough in the last question, and the coefficients of normal and frictional elasticity be the same for the table as for the cushion, and the ball be rolling without sliding when it impinges, the condition for the same thing will be

$$2aw = 7V \cos \alpha.$$

15. A billiard ball at rest on a table receives a blow in a given direction: supposing the table to be so inelastic that the ball does not rebound, prove that, if the coefficient of im-

pulsive friction $\mu = \sqrt{\frac{3}{7}}$, the ball will begin to roll without sliding, provided that the point where the ball is struck lies between two certain circles on the surface of the ball, whose planes are equidistant from its centre and one of them horizontal. Determine also the limits within which the blow must be delivered, in order that there may be no initial sliding, when μ is greater or less than $\sqrt{\frac{3}{7}}$.

If the ball be so struck that it begins to slide as well as roll, prove that, when its sliding motion is destroyed by friction, its ultimate direction and velocity will be the same as if the friction had been sufficient to cause it to roll without sliding at first: the coefficients of finite and impulsive friction being supposed to be equal.

16. Every particle of a sphere of radius a , which is placed on a fixed perfectly rough sphere of radius c , is attracted to a centre of force, on the surface of the fixed sphere, with a force varying inversely as the square of the distance; if it be placed at the extremity of the diameter through the centre of force and be then slightly displaced, determine its motion; and shew that when it leaves the fixed sphere the distance of its centre from the centre of force is a root of the equation

$$20x^3 - 13(2c + a)x^2 + 7a(2c + a)^2 = 0.$$

17. A rough cylinder of mass $2nm$, capable of motion about its horizontal axis, has a particle of mass m and coefficient of friction μ , placed on it vertically above the axis. The system is then disturbed. Find the motion and shew that the particle will slip on the cylinder after it has moved through an angle θ given by the equation

$$(n + 3)\mu \cos \theta - 2\mu - n \sin \theta = 0.$$

Find the subsequent motion until the particle leaves the cylinder.

CHAPTER VII.

ON CERTAIN GENERAL PRINCIPLES.

88. The product of the mass of any moving particle into its velocity is called the *momentum* of the particle.

The momentum of a body or collection of bodies is the sum of the momenta of all the particles of the system.

The momentum of a particle resolved in any direction is the product of the mass into the resolved part of the velocity in that direction.

The momentum of a body or system of bodies resolved in any direction is the sum of the momenta of all the particles resolved in that direction.

The *moment of the momentum* of a particle round any line is the product of the momentum of the particle resolved perpendicular to the line into the least distance between the line and the direction of motion of the particle. This is sometimes called the *angular momentum* of the particle about the line.

The *moment of momentum* of a body or system of bodies about any line is the sum of the moments of momentum of the several particles of the system about that line. This is sometimes called the *angular momentum* of the body about the line.

89. The first principle is that of *Conservation of Linear Momentum*, and may be enunciated as follows :

If a system of particles be acted on by forces the sum of whose resolved parts in any given direction vanishes, then the momentum of the system resolved in that direction is constant.

In the case supposed, if the given direction be taken as axis of x , we have by the first equation of (1) of Art. 23

$$\Sigma m \frac{d^2 x}{dt^2} = 0;$$

whence
$$\Sigma m \frac{dx}{dt} = \text{constant} \dots \dots \dots (1),$$

which proves the proposition.

The result of Art. 26 may be compared with this principle, and deduced from it as a particular case.

90. The second principle, that of *the Conservation of Moment of Momentum*, or, as it was formerly called, the *Conservation of Areas*, may be enunciated as follows :

If a system of particles be acted on by forces whose resultant has no moment round any fixed straight line, the moment of the momentum of the system round this line is constant.

For if this line be taken as the axis of z , by the last of the equations (2) of Art. 23,

$$\Sigma m \left(x \frac{d^2 y}{dt^2} - y \frac{d^2 x}{dt^2} \right) = 0;$$

whence, by integration,

$$\Sigma m \left(x \frac{dy}{dt} - y \frac{dx}{dt} \right) = \text{constant} = h_3 \dots \dots \dots (2).$$

Now the velocity of the particle m resolved perpendicular to the axis of z consists of $\frac{dx}{dt}$ parallel to Ox and $\frac{dy}{dt}$ parallel to Oy . The moments of these resolved parts round Oz are respectively $-y \frac{dx}{dt}$ and $x \frac{dy}{dt}$. Consequently $m \left(x \frac{dy}{dt} - y \frac{dx}{dt} \right)$ is the moment of the momentum of m round the axis of z ,

and $\Sigma m \left(x \frac{dy}{dt} - y \frac{dx}{dt} \right)$ is the moment of the momentum of the system round the axis of z .

In the application of both of these principles the internal actions between the different bodies of the system may be omitted from consideration in virtue of the general principle that the action and reaction between any two bodies whether tending to give them translation or rotation are equal and opposite. This will not be the case with regard to the next principle.

91. The product of the mass of a particle into the square of its velocity is called the *vis viva* of the particle. The half of the *vis viva* is called the *kinetic energy* of the particle. The *vis viva* or *kinetic energy* of a body or system of bodies is the sum of the *vires vivæ* or *kinetic energies* respectively of the several particles of the system.

Let us suppose that the forces acting on the several particles of a system depend only on the position of the particles and always have the same value when the particles return to the same positions. Let X, Y, Z be the resolved parts of the force acting on a particle of mass m whose co-ordinates are x, y, z parallel to the axes. Then we know that $Xdx + Ydy + Zdz$ is always an exact differential of some function of x, y, z . If we call this function $-U$, so that $Xdx + Ydy + Zdz = -dU$, mU is called the *potential energy* of the particle.

The potential energy of the body or system of bodies is the sum of the potential energies of the different particles.

92. The third principle, that of *Conservation of Energy*, may be enunciated as follows :

If a system of particles be in motion under the action of any conservative system of forces, that is of forces which depend only on the position of the particles, then throughout the motion the sum of the kinetic and potential energies of the system is constant.

This principle is sometimes called *the Principle of Vis Viva*.

93. By D'Alembert's principle the impressed forces acting on the particles of the body are in equilibrium with the reversed effective forces. Hence if δx , δy , δz be any displacements of the particle m at (x, y, z) parallel to the axes, consistent with the geometrical conditions of the system, by the Principle of Virtual Velocities we have

$$\Sigma m \left\{ \left(X - \frac{d^2x}{dt^2} \right) \delta x + \left(Y - \frac{d^2y}{dt^2} \right) \delta y + \left(Z - \frac{d^2z}{dt^2} \right) \delta z \right\} = 0 \dots (3),$$

since $m \left(X - \frac{d^2x}{dt^2} \right)$, $m \left(Y - \frac{d^2y}{dt^2} \right)$, $m \left(Z - \frac{d^2z}{dt^2} \right)$

are the forces acting on the particle m in the directions of the displacements δx , δy , δz .

Now among the possible values for δx , δy , δz are the actual displacements $\frac{dx}{dt} \delta t$, $\frac{dy}{dt} \delta t$, $\frac{dz}{dt} \delta t$ of the particle m during an indefinitely small time δt . If we consider them to have these values, we may deduce as one result of (3)

$$\Sigma m \left\{ \left(\frac{d^2x}{dt^2} - X \right) \frac{dx}{dt} + \left(\frac{d^2y}{dt^2} - Y \right) \frac{dy}{dt} + \left(\frac{d^2z}{dt^2} - Z \right) \frac{dz}{dt} \right\} \delta t = 0,$$

or, if v be the velocity and mU the potential energy of the particle m ,

$$\Sigma m \frac{d}{dt} \left(\frac{1}{2} v^2 + U \right) = 0 ;$$

whence, by integration,

$$\frac{1}{2} \Sigma m v^2 + \Sigma m U = \text{constant} \dots \dots \dots (4),$$

which proves the principle.

94. These principles give nothing beyond what may be deduced from the equations of motion of any body to which they apply. They often guide us to integrals of those equations which without their help might be difficult to find, and occasionally give us all the results we require. A large number of the forces acting on the bodies of the system will not come into either of these equations. It has been already remarked, that all the internal actions between the different parts of the system are to be omitted from consideration in

the equations of Conservation of Momentum and of Moment of Momentum. This is true even if impulsive actions or explosions and disruptions of a mechanical nature take place during the motion.

In the equation of the Conservation of Energy all forces may be omitted whose points of application either remain unchanged during the motion, or are always displaced in a direction perpendicular to the line of action of the force. For all such forces as these $X \frac{dx}{dt} + Y \frac{dy}{dt} + Z \frac{dz}{dt}$ obviously vanishes. All mutual actions between two parts of the system whose distances remain unaltered may also be omitted, for it is evident that the action and reaction will in this case give equal and opposite values of $X \frac{dx}{dt} + Y \frac{dy}{dt} + Z \frac{dz}{dt}$. Forces between mutually attracting particles whose distance can vary will obviously however form part of the potential energy.

95. In order to apply any of these principles to the solution of problems we must investigate the calculation of the kinetic energy and moment of momentum of a body moving in any manner.

We will first prove that the kinetic energy, and likewise the moment of momentum of any rigid body, consists of two parts, one due to the motion of translation of the centre of inertia and equal to what would be its value if the whole mass were collected at that point; the other due to rotation round that centre, whose value is the same as if that centre were stationary.

Let x, y, z be the co-ordinates of any point referred to axes fixed in space; $\bar{x}, \bar{y}, \bar{z}$ the co-ordinates of the centre of inertia referred to the same axes; x', y', z' the co-ordinates of the point (x, y, z) referred to parallel axes through the centre of inertia.

Then $x = \bar{x} + x', y = \bar{y} + y', z = \bar{z} + z'$.

Also $\sum m x' = 0, \sum m y' = 0, \sum m z' = 0 \dots\dots\dots(1),$

by the properties of the centre of inertia;

therefore $\sum m \frac{dx'}{dt} = 0, \sum m \frac{dy'}{dt} = 0, \sum m \frac{dz'}{dt} = 0 \dots\dots\dots(2).$

Hence the kinetic energy of the system

$$\begin{aligned}
 &= \frac{1}{2} \Sigma m \left\{ \left(\frac{dx}{dt} \right)^2 + \left(\frac{dy}{dt} \right)^2 + \left(\frac{dz}{dt} \right)^2 \right\} \\
 &= \frac{1}{2} \Sigma m \left\{ \left(\frac{d\bar{x}}{dt} + \frac{dx'}{dt} \right)^2 + \left(\frac{d\bar{y}}{dt} + \frac{dy'}{dt} \right)^2 + \left(\frac{d\bar{z}}{dt} + \frac{dz'}{dt} \right)^2 \right\} \\
 &= \frac{1}{2} \Sigma m \left\{ \left(\frac{d\bar{x}}{dt} \right)^2 + \left(\frac{d\bar{y}}{dt} \right)^2 + \left(\frac{d\bar{z}}{dt} \right)^2 \right\} \\
 &\quad + \frac{d\bar{x}}{dt} \Sigma m \frac{dx'}{dt} + \frac{d\bar{y}}{dt} \Sigma m \frac{dy'}{dt} + \frac{d\bar{z}}{dt} \Sigma m \frac{dz'}{dt} \\
 &\quad + \frac{1}{2} \Sigma m \left\{ \left(\frac{dx'}{dt} \right)^2 + \left(\frac{dy'}{dt} \right)^2 + \left(\frac{dz'}{dt} \right)^2 \right\} \\
 &= \frac{1}{2} \Sigma m \bar{v}^2 + \frac{1}{2} \Sigma m v'^2,
 \end{aligned}$$

if \bar{v} be the velocity of the centre of inertia, and v' the velocity of (x, y, z) relative to this point. This proves the proposition for the kinetic energy.

Again, the moment of momentum of the system round the axis of z

$$\begin{aligned}
 &= \Sigma m \left(x \frac{dy}{dt} - y \frac{dx}{dt} \right) \\
 &= \Sigma m \left\{ (\bar{x} + x') \left(\frac{d\bar{y}}{dt} + \frac{dy'}{dt} \right) - (\bar{y} + y') \left(\frac{d\bar{x}}{dt} + \frac{dx'}{dt} \right) \right\} \\
 &= \Sigma m \left(\bar{x} \frac{d\bar{y}}{dt} - \bar{y} \frac{d\bar{x}}{dt} \right) + \Sigma m \left(x' \frac{dy'}{dt} - y' \frac{dx'}{dt} \right) \\
 &\quad + \bar{x} \Sigma m \frac{dy'}{dt} + \frac{d\bar{y}}{dt} \Sigma m x' - \bar{y} \Sigma m \frac{dx'}{dt} - \frac{d\bar{x}}{dt} \Sigma m y' \\
 &= \Sigma m \left(\bar{x} \frac{d\bar{y}}{dt} - \bar{y} \frac{d\bar{x}}{dt} \right) + \Sigma m \left(x' \frac{dy'}{dt} - y' \frac{dx'}{dt} \right)
 \end{aligned}$$

by (1) and (2); which proves the proposition for the moment of momentum. These results may be compared with those of Art. 32.

96. The latter part of each of these expressions can be reduced by means of the formulæ (8) of Art. 7.

The portion of the kinetic energy due to rotation round the centre of inertia becomes thus,

$$\frac{1}{2} \Sigma m \{ (z' \omega_y - y' \omega_z)^2 + (x' \omega_z - z' \omega_x)^2 + (y' \omega_x - x' \omega_y)^2 \},$$

if $\omega_x, \omega_y, \omega_z$ be the angular velocities round the axes.

This reduces to

$$\begin{aligned} & \frac{1}{2} \{ \omega_x^2 \Sigma m (y'^2 + z'^2) + \omega_y^2 \Sigma m (z'^2 + x'^2) + \omega_z^2 \Sigma m (x'^2 + y'^2) \\ & \quad - 2 \omega_y \omega_z \Sigma m y' z' - 2 \omega_z \omega_x \Sigma m z' x' - 2 \omega_x \omega_y \Sigma m x' y' \} \\ &= \frac{1}{2} \{ A \omega_x^2 + B \omega_y^2 + C \omega_z^2 - 2 A' \omega_y \omega_z - 2 B' \omega_z \omega_x - 2 C' \omega_x \omega_y \}, \end{aligned}$$

with the notation of Art. 34,

$$= \frac{1}{2} I \omega^2 \text{ by the same Article,}$$

where I is the moment of inertia about the instantaneous axis and ω the resultant angular velocity, the direction-cosines of the instantaneous axis being $\frac{\omega_x}{\omega}, \frac{\omega_y}{\omega}, \frac{\omega_z}{\omega}$.

It follows from this result that equation (4) of Art. 71 is merely the expression of the principle of Conservation of Energy in that particular problem.

97. The portion of the moment of momentum round the axis of z due to rotation round the centre of inertia is

$$\begin{aligned} & \Sigma m \left(x' \frac{dy'}{dt} - y' \frac{dx'}{dt} \right) \\ &= \Sigma m \{ x' (x' \omega_z - z' \omega_x) - y' (z' \omega_y - y' \omega_z) \} \\ &= \omega_z \Sigma m (x'^2 + y'^2) - \omega_y \Sigma m y' z' - \omega_x \Sigma m z' x' \\ &= C \omega_z - A' \omega_y - B' \omega_x, \end{aligned}$$

and similar results for the moment of momentum round the other axes.

If the axes be principal axes the moments of momentum round the axes reduce to $A \omega_x, B \omega_y$ and $C \omega_z$ respectively.

98. If l, m, n be the direction-cosines of any line OP , the moment of momentum of the body round this line will be equal to $l h_1 + m h_2 + n h_3$, where h_1, h_2, h_3 are the moments of momentum round the axes. This follows from the fact

that the moment of momentum of a particle of mass μ , round any line as the axis of z , can be expressed as $2\mu \frac{dA}{dt}$, where A is the sectorial area described by the projection of the particle on the plane of xy round the origin. Thus the portions of $h_1\delta t$, $h_2\delta t$, $h_3\delta t$ arising from the motion of this particle are ultimately the products of μ into the projections on the co-ordinate planes of the area of the triangle whose angular points are the origin and two consecutive positions of the particle, and the projection of this triangle on the plane perpendicular to the line OP multiplied by μ , will give the corresponding terms in $(lh_1 + mh_2 + nh_3)\delta t$.

99. The result of the last article can also be proved analytically in the following manner.

Let a new set of fixed rectangular co-ordinate axes be taken, of which OP is one; and let l', m', n' ; l'', m'', n'' be the direction-cosines of the other two. Let x', y', z' be the co-ordinates of the point (x, y, z) with reference to these new axes.

$$\begin{aligned}\text{Hence} \quad y' &= l'x + m'y + n'z \\ z' &= l''x + m''y + n''z.\end{aligned}$$

Hence

$$\begin{aligned}y' \frac{dz'}{dt} - z' \frac{dy'}{dt} &= (l'x + m'y + n'z) \left(l'' \frac{dx}{dt} + m'' \frac{dy}{dt} + n'' \frac{dz}{dt} \right) \\ &\quad - (l''x + m''y + n''z) \left(l' \frac{dx}{dt} + m' \frac{dy}{dt} + n' \frac{dz}{dt} \right) \\ &= (l'm'' - l''m') \left(x \frac{dy}{dt} - y \frac{dx}{dt} \right) \\ &\quad + (m'n'' - m''n') \left(y \frac{dz}{dt} - z \frac{dy}{dt} \right) + (n'l'' - n''l') \left(z \frac{dx}{dt} - x \frac{dz}{dt} \right) \\ &= n \left(x \frac{dy}{dt} - y \frac{dx}{dt} \right) + l \left(y \frac{dz}{dt} - z \frac{dy}{dt} \right) + m \left(z \frac{dx}{dt} - x \frac{dz}{dt} \right)\end{aligned}$$

by means of the usual relations between $l, m, n, l', m', n', l'', m'', n''$.

This proves the proposition for one particle, and by summation it follows for the whole system.

100. If a body be moving about a fixed point and h_1, h_2, h_3 be at any time the moments of momentum about the co-ordinate axes, the equations of motion can be written

$$\frac{dh_1}{dt} = L, \quad \frac{dh_2}{dt} = M, \quad \frac{dh_3}{dt} = N.$$

If the forces acting have no moment about the fixed point so that L, M, N all vanish, it follows that h_1, h_2, h_3 are all constant in value. Thus the moment of momentum round any line fixed in space will be also constant.

If we assume a quantity $H = \sqrt{h_1^2 + h_2^2 + h_3^2}$, there will be a line whose direction-cosines are $\frac{h_1}{H}, \frac{h_2}{H}, \frac{h_3}{H}$, and if θ be the angle between this line and any other line whose direction-cosines are l, m, n ; the moment of momentum of the body round this second line $= lh_1 + mh_2 + nh_3$
 $= H \cos \theta$.

Hence the moment of momentum round the line (l, m, n) is always less than H except when $\theta = 0$. The line whose direction-cosines are proportional to h_1, h_2, h_3 has therefore the property that the moment of momentum round it is greater than that round any other axis.

It is obvious firstly that this line is fixed throughout the motion, and secondly that it is the same line whatever axes may be taken.

101. If the axes at any instant coincide with the principal axes at the fixed point, by Art. 97, $h_1 = A\omega_x$, $h_2 = B\omega_y$, $h_3 = C\omega_z$. Hence the line round which the moment of momentum is a maximum, coincides with the invariable line of Arts. 72, 73. We thus obtain another proof of the invariability of this line in space.

We see also that equation (5) of Art. 71 is merely the expression of the result given by the law of Conservation of Moment of Momentum.

102. The proposition of Art. 100 applies to the motion of a system of bodies acted on only by their mutual attractions or repulsions. In every such system by Art. 26, the centre of inertia is either fixed or moves with uniform velocity in a straight line.

If this point be taken as origin the equations of motion are the same in form as if it were a fixed point, and in virtue of the principle that action and reaction, whether they are linear or angular in their mechanical tendencies, are always equal and opposite, L , M and N all vanish for the whole system. Hence the moment of momentum of the system round any line through the centre of inertia remains constant through the motion: and there is one line round which this moment is greater than round any other. This line is called the invariable line of the system, and a plane through the centre of inertia perpendicular to it is called the invariable plane.

103. The principles of Conservation of Momentum and of Moment of Momentum hold good even if there be any impulsive actions, or if any explosive actions, entirely arising within it, take place between the parts of the system during the motion.

104. In the case of a single body moving in a plane the expressions for the kinetic energy and moment of momentum are somewhat simplified.

Let us take the plane of motion as plane of xy and let \bar{x} , \bar{y} be the co-ordinates of the centre of inertia. Let also θ be the angle which some line fixed in the body makes with the axis of x , M the mass of the body, and Mk^2 its moment of inertia round a line through the centre of inertia perpendicular to the plane of xy , which, if the body be a plane lamina, or symmetrical with respect to the plane of xy , is a principal axis (Arts. 38, 40). Then, by Articles 95 and 96, the kinetic energy of the body can be expressed as

$$\frac{1}{2}M \left\{ \left(\frac{d\bar{x}}{dt} \right)^2 + \left(\frac{d\bar{y}}{dt} \right)^2 \right\} + \frac{1}{2}Mk^2 \left(\frac{d\theta}{dt} \right)^2 \dots\dots\dots(1),$$

while, by Articles 95 and 97, the moment of momentum round a line perpendicular to the plane of xy will be

$$M \left(\bar{x} \frac{d\bar{y}}{dt} - \bar{y} \frac{d\bar{x}}{dt} \right) + Mk^2 \frac{d\theta}{dt} \dots\dots\dots(2).$$

If the position of the centre of inertia be given by polar co-ordinates r and ϕ , we know that

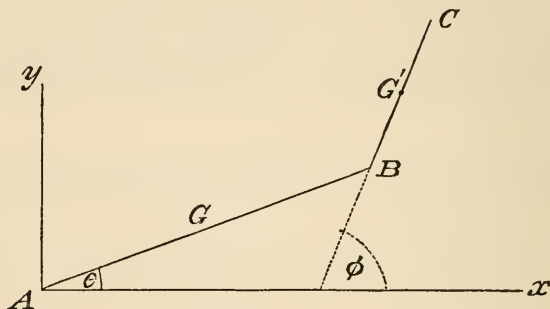
$$\left(\frac{d\bar{x}}{dt} \right)^2 + \left(\frac{d\bar{y}}{dt} \right)^2 = \left(\frac{dr}{dt} \right)^2 + r^2 \left(\frac{d\phi}{dt} \right)^2 \dots\dots\dots(3),$$

$$\bar{x} \frac{d\bar{y}}{dt} - \bar{y} \frac{d\bar{x}}{dt} = r^2 \frac{d\phi}{dt} \dots\dots\dots(4),$$

which give us other forms, sometimes more convenient, for the kinetic energy and moment of momentum.

105. As an instance of the application of these principles let us take the following problem.

Two uniform rods AB , BC jointed at B can rotate freely in a horizontal plane about the end A which is fixed. It is required to find the positions and velocities of the rods at any time.



Let AB , BC make angles θ , ϕ with some fixed line Ax at the time t , and let x' , y' be the co-ordinates of the middle point of BC , referred to Ax and a line perpendicular to it through A as axes. Let $AB = 2a$, $BC = 2b$; let M , M' be the masses of the rods, and k , k' their radii of gyration about their centres of inertia. Then the actions of the rods on each other at B , and the action of the fixed point at A will not affect either the energy or the moment of momentum. Hence by the principle of Conservation of Energy,

$$M(a^2 + k^2) \left(\frac{d\theta}{dt} \right)^2 + M' \left\{ \left(\frac{dx'}{dt} \right)^2 + \left(\frac{dy'}{dt} \right)^2 \right\} + M'k'^2 \left(\frac{d\phi}{dt} \right)^2 = \text{constant}.$$

Also by the Conservation of Moment of Momentum,

$$M(a^2 + k^2) \frac{d\theta}{dt} + M' \left\{ x' \frac{dy'}{dt} - y' \frac{dx'}{dt} \right\} + M'k'^2 \frac{d\phi}{dt} = \text{constant}.$$

The values of the constants must be determined from initial circumstances.

From the geometry we obtain

$$x' = 2a \cos \theta + b \cos \phi,$$

$$y' = 2a \sin \theta + b \sin \phi,$$

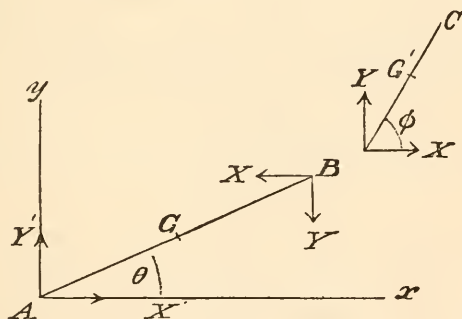
whence

$$\left(\frac{dx'}{dt}\right)^2 + \left(\frac{dy'}{dt}\right)^2 = 4a^2 \left(\frac{d\theta}{dt}\right)^2 + b^2 \left(\frac{d\phi}{dt}\right)^2 + 4ab \cos(\phi - \theta) \frac{d\theta}{dt} \frac{d\phi}{dt}.$$

$$x' \frac{dy'}{dt} - y' \frac{dx'}{dt} = 4a^2 \frac{d\theta}{dt} + b^2 \frac{d\phi}{dt} + 2ab \cos(\phi - \theta) \left(\frac{d\theta}{dt} + \frac{d\phi}{dt}\right).$$

Thus the two equations above enable us to determine $\frac{d\theta}{dt}$ and $\frac{d\phi}{dt}$ in terms of constants and $(\phi - \theta)$.

106. It may not be without value to the student to solve the above problem by the use of the general equations of motion.



We may assume that the action of the fixed point A on AB has for components parallel to the axes X' , Y' , and that the action of AB on BC at B has for components in the same directions X , Y . The action of BC on AB is exactly equal and opposite to that of AB on BC . Let x , y be the co-ordinates of the centre of inertia of AB , and let other quantities be denoted as in the last article. Then by equations (2) of Art. 79, we have for the motion of the centre of inertia of AB ,

$$M \frac{d^2 x}{dt^2} = X' - X \dots \dots \dots (1),$$

$$M \frac{d^2 y}{dt^2} = Y' - Y \dots \dots \dots (2),$$

and by equation (1) of the same article

$$Mk^2 \frac{d^2\theta}{dt^2} = (X + X') a \sin \theta - (Y + Y') a \cos \theta \dots\dots(3).$$

The geometrical relations give

$$x = a \cos \theta, \quad y = a \sin \theta \dots\dots\dots(4).$$

We may notice that by equation (6) of Art. 52, since A is a fixed point we can at once write down the equation

$$M(a^2 + k^2) \frac{d^2\theta}{dt^2} = Xa \sin \theta - Ya \cos \theta \dots\dots\dots(5),$$

which may replace (1), (2) and (3) if we do not wish to find the values of X' and Y' . Equation (5) can also be obtained from (1), (2) and (3) by eliminating X' and Y' since by means of (4) it is easily proved that

$$\frac{d^2y}{dt^2} \cos \theta - \frac{d^2x}{dt^2} \sin \theta = a \frac{d^2\theta}{dt^2}.$$

For the motion of BC , we similarly have

$$M' \frac{d^2x'}{dt^2} = X \dots\dots\dots(6),$$

$$M' \frac{d^2y'}{dt^2} = Y \dots\dots\dots(7),$$

$$M'k'^2 \frac{d^2\phi}{dt^2} = Xb \sin \phi - Yb \cos \phi \dots\dots(8),$$

and for geometrical relations

$$x' = 2a \cos \theta + b \cos \phi, \quad y' = 2a \sin \theta + b \sin \phi \dots\dots(9).$$

107. If we multiply (1) by $\frac{dx}{dt}$, (2) by $\frac{dy}{dt}$, (3) by $\frac{d\theta}{dt}$, (6), (7) and (8) by $\frac{dx'}{dt}$, $\frac{dy'}{dt}$ and $\frac{d\phi}{dt}$ respectively and add all the results, the coefficients of X' , Y' , X , and Y will vanish identically by means of (4) and (9). The coefficient of X for instance is

$$-\frac{dx}{dt} + a \sin \theta \frac{d\theta}{dt} + \frac{dx'}{dt} + b \sin \phi \frac{d\phi}{dt}$$

$$= a \sin \theta \frac{d\theta}{dt} + a \sin \theta \frac{d\theta}{dt} - 2a \sin \theta \frac{d\theta}{dt} - b \sin \phi \frac{d\phi}{dt} + b \sin \phi \frac{d\phi}{dt} = 0,$$

by differentiating the first equations of (4) and (9).

Hence

$$\left\{ \frac{d^2 x}{dt^2} \frac{dx}{dt} + \frac{d^2 y}{dt^2} \frac{dy}{dt} + k^2 \frac{d^2 \theta}{dt^2} \frac{d\theta}{dt} \right\} + M' \left\{ \frac{d^2 x'}{dt^2} \frac{dx'}{dt} + \frac{d^2 y'}{dt^2} \frac{dy'}{dt} + k'^2 \frac{d^2 \phi}{dt^2} \frac{d\phi}{dt} \right\} = 0.$$

Therefore integrating

$$I \left\{ \left(\frac{dx}{dt} \right)^2 + \left(\frac{dy}{dt} \right)^2 + k^2 \left(\frac{d\theta}{dt} \right)^2 \right\} + \frac{1}{2} M' \left\{ \left(\frac{dx'}{dt} \right)^2 + \left(\frac{dy'}{dt} \right)^2 + k'^2 \left(\frac{d\phi}{dt} \right)^2 \right\} = \text{constant}.$$

which is the same as the first equation of Art. 105.

Multiplying (1) by $-y$, (2) by x , and adding to (3) we get in virtue of (4)

$$M \left\{ x \frac{d^2 y}{dt^2} - y \frac{d^2 x}{dt^2} \right\} + k^2 \frac{d^2 \theta}{dt^2} = X \cdot 2a \sin \theta - Y \cdot 2a \cos \theta.$$

Treating (6), (7) and (8) similarly, we get by means of (9),

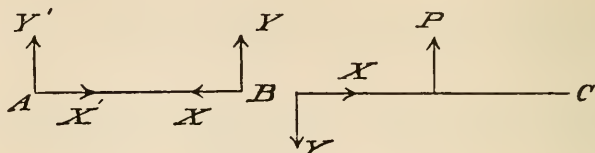
$$M' \left\{ x' \frac{d^2 y'}{dt^2} - y' \frac{d^2 x'}{dt^2} \right\} + k'^2 \frac{d^2 \phi}{dt^2} = Y \cdot 2a \cos \theta - X \cdot 2a \sin \theta.$$

And by adding these and integrating we get the second equation of Art. 105. This problem may serve as an example of the method of deducing the equations of conservation of energy and moment of momentum from the regular equations of motion in any cases in which the principles apply.

108. It only remains to show how the constants may be determined from initial conditions.

Suppose as an instance that the rods are originally lying at rest in a straight line, and that an impulse P is applied to the middle point of BC at right angles to its length. There will be a sudden impulsive action between AB and BC at B and also a sudden impulsive action at A on AB . Let these act as represented in the figure. Let u' , v' be the initial

velocities of the centre of inertia of BC along and perpendicular to BC , u , v those of the centre of inertia of AB , and let ω , ω' be the initial angular velocities of AB and BC . Then by Articles 28 and 64 we have for the motion of AB ,



$Mu = X' - X$, $Mv = Y + Y'$, $Mk^2\omega = (Y - Y')a \dots (1)$;
and for the motion of BC ,

$$M'u' = X$$
, $M'v' = P - Y$, $M'k'^2\omega' = Yb \dots \dots \dots (2)$.

Also the velocity of the point A vanishes, whence by Art. 8,

$$u = 0 \quad v - a\omega = 0 \dots \dots \dots (3).$$

And the velocity of the point B in AB is the same as that of the point B of BC , whence

$$u = u', \quad v + a\omega = v' - b\omega' \dots \dots \dots (4).$$

These equations give at once

$$u = 0, \quad u' = 0, \quad X = 0, \quad X' = 0,$$

and by substituting for v , v' , ω , ω' their values from (1) and (2) in (3) and (4) we get

$$Y + Y' - \frac{a^2}{k^2} (Y - Y') = 0$$

$$\frac{Y + Y' + \frac{a^2}{k^2} (Y - Y')}{M} = \frac{P - Y - \frac{b^2}{k'^2} Y}{M'},$$

whence Y and Y' are found, and then ω and ω' which are the initial values of $\frac{d\theta}{dt}$ and $\frac{d\phi}{dt}$. The constants on the right-hand sides of the equations in Art. 105 can then be determined.

A similar investigation will determine the angular velocities if the rods be originally moving in any manner and strike against a fixed obstacle; or if one point becomes fixed. The reactions at the fixed point are unknown, but the geometrical condition of the point being fixed will supply the other equations necessary for determining them.

EXAMPLES. CHAPTER VII.

1. A straight tube of given length is capable of turning about one extremity in a horizontal plane; a particle of mass one-third that of the tube is placed at a given point within it at rest, an angular velocity is given to the system, determine the velocity of the particle on leaving the tube.

2. A horizontal tube is rotating about its middle point in a horizontal plane in which it is constrained to move; a rod of equal length and mass is shot into it; determine the initial velocity of the rod that its middle point may just reach that of the tube; and during the motion determine at what point the resultant action between the two acts.

3. A heavy circular disc is revolving in a horizontal plane about its centre, which is fixed. An insect walks from the centre uniformly along a certain radius, and then flies away. Determine the whole motion.

4. A circular disc is moving uniformly with an angular velocity Ω , about an axis through its centre perpendicular to its plane. An insect alights on the edge and crawls along a curve drawn on the disc in the form of a lemniscate, with uniform relative angular velocity α , the curve touching the circle. If $\alpha = \frac{1}{3} \Omega$, and mass of insect $= \frac{1}{16}$ mass of disc, then the angle turned through by the disc when the insect arrives at the centre is equal to

$$\frac{24}{\sqrt{7}} \tan^{-1} \frac{1}{3} - \frac{\pi}{4}.$$

5. A rough horizontal plane lamina is capable of rotating freely round a vertical axis. If a heavy particle of mass m be placed at any point upon it, and an angular velocity ω be given to the plane, show that the length of the arc traversed by the particle on the plane when it just comes to rest relatively to the plane will be $\frac{Mk^2}{2\mu mg} \omega (\omega - \omega')$; ω' being the ultimate angular velocity, Mk^2 the moment of inertia of the lamina about the axis of revolution, and μ the coefficient of friction.

6. Three equal rods placed in a straight line are jointed by hinges to one another, they move with a velocity v perpendicular to their lengths: if the middle point of the middle one become suddenly fixed, show that the extremities of the other two will meet in time $\frac{4\pi a}{9v}$, a being the length of each rod.

7. Three equal uniform inelastic rods loosely jointed together are laid in a straight line on a smooth horizontal table, and the two outer ones are set in motion about the ends of the middle one with equal angular velocities (1) in the same direction, (2) in opposite directions, prove that, in the first case, when the outer rods make the greatest angle with the direction of the middle one produced on each side, the common angular velocity of the three is $\frac{4\omega}{7}$, and in the second case, that after the impact of the two outer rods, the triangle formed by them will move with uniform velocity equal to $\frac{2a\omega}{3}$, $2a$ being the length of each rod.

8. AB, BC are two equal uniform rods in a straight line, loosely jointed at B and moving with the same velocity in a direction perpendicular to their length; if the end A be suddenly fixed, show that the initial angular velocity of AB is three times that of BC . Also show that in the subsequent motion of the rods the greatest angle between them equals $\cos^{-1} \frac{2}{3}$, and that, when they are next in a straight line, the angular velocity of BC is nine times that of AB .

9. A wire in the form of a circle is capable of revolving in a horizontal plane about a fixed point in its circumference, and two small rings attached to the ends of a rod slide upon it; the wire is set rotating with a given angular velocity: determine the subsequent motion.

10. A square, formed of four similar uniform rods jointed freely at their extremities, is laid upon a smooth horizontal table, one of its angular points being fixed; if angular velocities ω , ω' in the plane of the table be communicated to the two sides containing this angle, show that the greatest value of the angle (2α) between them is given by the equation, $\cos 2\alpha = -\frac{5}{6} \frac{(\omega - \omega')^2}{\omega^2 + \omega'^2}$.

11. A rectangle is formed of four uniform rods of lengths $2a$ and $2b$ respectively, which are connected by hinges at their ends. The rectangle is revolving about its centre on a smooth horizontal plane with an angular velocity ω , when a point in one of the sides of length $2a$ suddenly becomes fixed. Show that the angular velocity of the sides of length $2b$ immediately becomes $\frac{3a+b}{6a+4b} \omega$. Find also the change in the angular velocity of the other sides, and the impulsive action at the point which becomes fixed.

12. A uniform rod is moveable freely about one end on a horizontal table, and the other is fastened to a particle of equal mass by a string of equal length with the rod. Initially the rod and string are in one straight line, the particle at rest, and the rod has an angular velocity given to it. Show that when they are again in a straight line, the angular velocity of the string is to that of the rod as 5 : 4 and the greatest angle between the string and rod is $\cos^{-1} \frac{\sqrt{106-5}}{9}$.

13. Two uniform rods OA , AB of lengths $2a$, $2b$ and of masses proportional to their lengths are hinged at A and are rotating round the fixed hinge O in the same straight line and with the same angular velocity when the outer AB comes against an obstacle P . If the position of this be such

as to reduce both rods to instantaneous rest, prove that

$$AP = 2b^2 \cdot \frac{3a + 2b}{2a^2 + 6ab + 3b^2}.$$

14. A system consisting of two straight rods rigidly connected together at their points of intersection is moving in its own plane so that each rod is in contact with one of two smooth pegs. Prove that no impulse which acts at the point of intersection and towards the centre of the circle round the point of intersection and the pegs can have any effect on the motion.

15. A uniform rod of length $2a$ can turn freely about one extremity. In its initial position it makes an angle of 90° with the vertical, and is projected horizontally with angular velocity ω : show that the least angle it makes with the vertical is given by the equation

$$4a\omega^2 \cos \theta = 3g \sin^2 \theta.$$

16. A rod of length $2a$ moveable about its lower end is inclined at an angle α to the vertical, and it is given a rotation ω about the vertical; if θ be its inclination to the vertical when its angular velocity about a horizontal axis is a maximum, show that

$$\sin^3 \theta \cdot \tan \theta + \frac{4}{3} \frac{a}{g} \omega^2 \sin^4 \alpha = 0.$$

17. A body whose centre of gravity G is fixed receives a blow P , the direction-cosines of which are $\frac{\sqrt{B-A}}{\sqrt{C-A}}$, 0 , $\frac{\sqrt{C-B}}{\sqrt{C-A}}$; A, B, C being the principal moments at the centre of gravity of the body. Prove that the whole *vis viva* generated varies as the square of the perpendicular from G on the direction of P .

18. Two uniform unequal rods AB, BC are hinged at B and supported in one vertical line so as just to touch a smooth horizontal plane at C . The support is withdrawn. Find the motion, and show that when they are both horizontal the distance through which the centre of BC has moved horizontally is

$$\frac{\text{mass } AB}{\text{sum of masses}} \times \text{half the difference between the lengths.}$$

CHAPTER VIII.

MISCELLANEOUS PROBLEMS.

109. WE have referred all the motions considered, to axes fixed in space, with the exception of those in Articles 66—77, where the co-ordinate axes revolved with the body. Some problems are simplified by referring the motions to axes which move in a manner independent of the motion of the body, and we proceed to investigate a few of the principal formulæ relating to such axes.

110. When the co-ordinates of a point referred to fixed axes are x, y, z , the velocities of the point parallel to these axes are $\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}$. This will no longer be the case if the axes are moving.

Let $l_1, m_1, n_1; l_2, m_2, n_2; l_3, m_3, n_3$ be the direction-cosines of the moving axes with reference to axes fixed in space; let ξ, η, ζ be the co-ordinates of a point referred to the moving axes; x, y, z the co-ordinates of the same point referred to fixed axes.

Then

$$x = l_1\xi + l_2\eta + l_3\zeta$$

$$y = m_1\xi + m_2\eta + m_3\zeta$$

$$z = n_1\xi + n_2\eta + n_3\zeta,$$

whence

$$\begin{aligned} \frac{dx}{dt} &= l_1 \frac{d\xi}{dt} + l_2 \frac{d\eta}{dt} + l_3 \frac{d\zeta}{dt} + \xi \frac{dl_1}{dt} + \eta \frac{dl_2}{dt} + \zeta \frac{dl_3}{dt} \\ &= l_1 \frac{d\xi}{dt} + l_2 \frac{d\eta}{dt} + l_3 \frac{d\zeta}{dt} - \theta_3 y + \theta_2 z, \end{aligned}$$

by a reduction identical with that of Art. 7, if $\theta_1, \theta_2, \theta_3$ be taken to denote the angular velocities of the system of moving axes round the axes of x, y, z respectively. Now if we suppose the fixed axes so taken as to coincide at the instant with the moving axes, the above equation gives us

$$\frac{dx}{dt} = \frac{d\xi}{dt} - \theta_3\eta + \theta_2\zeta,$$

that is the velocities parallel to the instantaneous positions of the moving axes are represented by

$$\frac{d\xi}{dt} - \theta_3\eta + \theta_2\zeta, \quad \frac{d\eta}{dt} - \theta_1\zeta + \theta_3\xi, \quad \frac{d\zeta}{dt} - \theta_2\xi + \theta_1\eta,$$

where $\theta_1, \theta_2, \theta_3$ are the angular velocities of the system of moving axes round lines which at any instant coincide with the moving axes, or, in other words, the angular velocities of the system round themselves.

111. We have shown in Art. 67 that if ω_x and ω_1 be angular velocities of a rigid body round lines, one fixed in space and the other fixed in the body, which at any instant coincide, not only is ω_1 equal to ω_x , which is obvious, but also $\frac{d\omega_x}{dt}$ is equal to $\frac{d\omega_1}{dt}$.

Let $\omega_x, \omega_y, \omega_z$ be the angular velocities of the body at any instant round axes fixed in space, $\omega_1, \omega_2, \omega_3$ the angular velocities round axes which are themselves revolving with angular velocities $\theta_1, \theta_2, \theta_3$ about themselves, we have with the notation of the last article,

$$\omega_x = l_1\omega_1 + l_2\omega_2 + l_3\omega_3.$$

Therefore

$$\begin{aligned} \frac{d\omega_x}{dt} &= l_1 \frac{d\omega_1}{dt} + l_2 \frac{d\omega_2}{dt} + l_3 \frac{d\omega_3}{dt} + \omega_1 \frac{dl_1}{dt} + \omega_2 \frac{dl_2}{dt} + \omega_3 \frac{dl_3}{dt} \\ &= l_1 \frac{d\omega_1}{dt} + l_2 \frac{d\omega_2}{dt} + l_3 \frac{d\omega_3}{dt} - \phi_3\omega_y + \phi_2\omega_z, \end{aligned}$$

by the same reduction as before, if ϕ_2, ϕ_3 be the angular velocities of the moving set of axes about the fixed axes of

y and z . If now at the instant the moving axes coincide with the fixed axes,

$$\frac{d\omega_x}{dt} = \frac{d\omega_1}{dt} - \theta_3\omega_2 + \theta_2\omega_3.$$

Similarly
$$\frac{d\omega_y}{dt} = \frac{d\omega_2}{dt} - \theta_1\omega_3 + \theta_3\omega_1,$$

$$\frac{d\omega_z}{dt} = \frac{d\omega_3}{dt} - \theta_2\omega_1 + \theta_1\omega_2.$$

The student will see that these include the result of Art. 67, in which case $\theta_1 = \omega_1$, $\theta_2 = \omega_2$, $\theta_3 = \omega_3$.

112. The results of these two articles can be obtained geometrically more briefly.

Let P be a point whose co-ordinates referred to the moving axes are at any instant ξ, η, ζ . Let P' be its position after a time δt and let $\xi + \delta\xi, \eta + \delta\eta, \zeta + \delta\zeta$ be the co-ordinates of P' . Let P'' be a point whose co-ordinates referred to the moving axes after the interval of time δt are ξ, η, ζ . Then the projection of PP' on any line, as the axis of x in its original position, is the sum of the projections of PP'' and $P''P'$. The projection of $P''P'$ on this line is ultimately $\delta\xi$, while that of PP'' , which is simply the displacement of a point moving rigidly connected with the axes, is by Art. 7, equal to $(\theta_2\zeta - \theta_3\eta) \delta t$. Hence the whole projection of PP' on the axis of x is ultimately $\delta\xi + (\theta_2\zeta - \theta_3\eta) \delta t$, and the velocity of P parallel to Ox is $\frac{d\xi}{dt} + \theta_2\zeta - \theta_3\eta$; similarly the velocities in the other directions can be obtained.

We have seen in Art. 14 that all propositions about the composition of linear velocities parallel to given lines, hold also in regard to the composition of angular velocities about those lines, whence the results of Art. 111 follow from those of 110.

113. The relations of Art. 111 can be used to modify the equations of Art. 68 in one case of not infrequent occurrence, namely, when two of the principal moments of inertia, as A and B , at the fixed point are equal. In this case the momental ellipsoid becomes a spheroid and any axis

whatever in the plane of AB is a principal axis. The two axes of A and B may therefore rotate in any manner in their own plane without altering the conditions on which the equations of Art. 68 are obtained. The only alterations we shall have to make, besides putting $B = A$, will be to write $\frac{d\omega_1}{dt} - \theta_3\omega_2$ for $\frac{d\omega_1}{dt}$ and $\frac{d\omega_2}{dt} + \theta_3\omega_1$ for $\frac{d\omega_2}{dt}$, where θ_3 is the angular velocity of the axes of A and B round that of C . The equations of motion thus become

$$\begin{aligned} A \left(\frac{d\omega_1}{dt} - \theta_3\omega_2 \right) + (C - A) \omega_2\omega_3 &= L \\ A \left(\frac{d\omega_2}{dt} + \theta_3\omega_1 \right) + (A - C) \omega_3\omega_1 &= M \\ C \frac{d\omega_3}{dt} &= N. \end{aligned}$$

One assumption, sometimes advantageous, is that $\theta_3 = -\omega_3$. The equations then reduce to

$$\begin{aligned} A \frac{d\omega_1}{dt} + C \omega_2\omega_3 &= L \\ A \frac{d\omega_2}{dt} - C \omega_1\omega_3 &= M \\ C \frac{d\omega_3}{dt} &= N. \end{aligned}$$

The discussion of the applications of the theory of moving axes is however beyond the limits of an introductory treatise and must be sought for in the larger works on the subject.

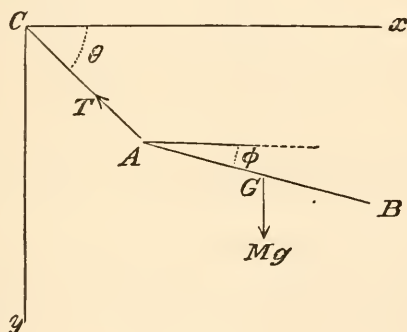
114. There are two classes of problems in which we do not require the complete solution of the equations of motion, but merely the determination of the values of certain quantities involved in them at certain times or under certain limitations as to magnitude: the problems of what are called "initial motions" and "small oscillations."

115. Each of these classes will be best illustrated by the actual solution of a problem, and an exceedingly simple one will do for the purpose.

Suppose a bar AB hangs horizontally, suspended by two equal vertical strings fastened to the ends. One of the strings is cut. It is required to find the instantaneous alteration in the tension of the other.

In this case it is not difficult, and it may be useful to the student, to write down the equations of motion corresponding to the position at a time t after the string has been cut.

Let then CA be the remaining string, of length l . Take C as origin, and horizontal and vertical lines through C as axes of x and y . Let the length of AB be $2a$, its mass M , and x, y the co-ordinates of its middle point. Let θ and ϕ be the inclinations of CA and AB respectively to the horizon. Let T be the tension of AC .



Then for the motion of AB , we have by Art. 79,

$$M \frac{d^2x}{dt^2} = -T \cos \theta \dots\dots\dots(1),$$

$$M \frac{d^2y}{dt^2} = Mg - T \sin \theta \dots\dots\dots(2),$$

$$Mk^2 \frac{d^2\phi}{dt^2} = Ta \sin (\theta - \phi) \dots\dots\dots(3).$$

The geometrical conditions give

$$x = l \cos \theta + a \cos \phi \dots\dots\dots(4),$$

$$y = l \sin \theta + a \sin \phi \dots\dots\dots(5).$$

We thus have five equations to determine the five quantities x, y, θ, ϕ, T .

One integral relation independent of T could be obtained by multiplying the first three equations by $\frac{dx}{dt}$, $\frac{dy}{dt}$ and $\frac{d\phi}{dt}$ respectively; adding the resulting equations, the coefficient of T would be found to vanish. This would give us the equation of energy. (Arts. 92--94.)

To find another integral relation independent of T , some special artifice would have to be adopted. We do not however want to solve the problem completely but only to discover the initial value of T .

If we differentiate each of the equations (4) and (5) twice with respect to t , we obtain

$$\begin{aligned}\frac{d^2x}{dt^2} &= -l \sin \theta \frac{d^2\theta}{dt^2} - a \sin \phi \frac{d^2\phi}{dt^2} - l \cos \theta \left(\frac{d\theta}{dt}\right)^2 - a \cos \phi \left(\frac{d\phi}{dt}\right)^2, \\ \frac{d^2y}{dt^2} &= l \cos \theta \frac{d^2\theta}{dt^2} + a \cos \phi \frac{d^2\phi}{dt^2} - l \sin \theta \left(\frac{d\theta}{dt}\right)^2 - a \sin \phi \left(\frac{d\phi}{dt}\right)^2.\end{aligned}$$

At the very beginning of the motion $\frac{d\theta}{dt}$ and $\frac{d\phi}{dt}$ both vanish, and θ and ϕ have the values $\frac{\pi}{2}$ and zero respectively. Therefore *initially*, that is, at the very instant when the string is cut,

$$\frac{d^2x}{dt^2} = -l \frac{d^2\theta}{dt^2}, \quad \frac{d^2y}{dt^2} = a \frac{d^2\phi}{dt^2}.$$

Hence equations (1), (2) and (3) become

$$-Ml \frac{d^2\theta}{dt^2} = 0, \quad Ma \frac{d^2\phi}{dt^2} = Mg - T, \quad Mk^2 \frac{d^2\phi}{dt^2} = Ta.$$

From the last two of which

$$T = \frac{Mg}{1 + \frac{k^2}{a^2}} = \frac{Mg}{4}, \text{ since } k^2 = \frac{a^2}{3}. \quad (\text{Art. 41.})$$

Hence since the tension of the string was $\frac{Mg}{2}$ before the other one was cut, it is diminished by one-half.

116. Any other similar problem can be treated in the same way. The dynamical equations need however only be written down with the values of the right hand members in the initial positions. The geometrical equations must be written down for the general position and differentiated twice. In the results thus obtained the initial values of the geometrical quantities can be substituted, those of the first differential coefficients being zero. The second differential coefficients can then be eliminated between the geometrical and dynamical results and the initial values of the reactions and tensions obtained.

117. The general equations of motion of a body or system of bodies always contain the second differential coefficients of the quantities which serve to determine the position of the bodies. We may call these quantities the *co-ordinates* of the bodies. From the equations of motion we can deduce the values of these co-ordinates which correspond to the position of equilibrium, by equating the accelerations to zero. If we assume that the co-ordinates have values slightly differing from those which give equilibrium, we can obtain a series of differential equations of the second order, and linear as regards the increments of the co-ordinates, any term involving any one of them being expanded as far as the first power of the increment. If between these equations we eliminate the unknown reactions, we shall arrive at a series of equations of the form

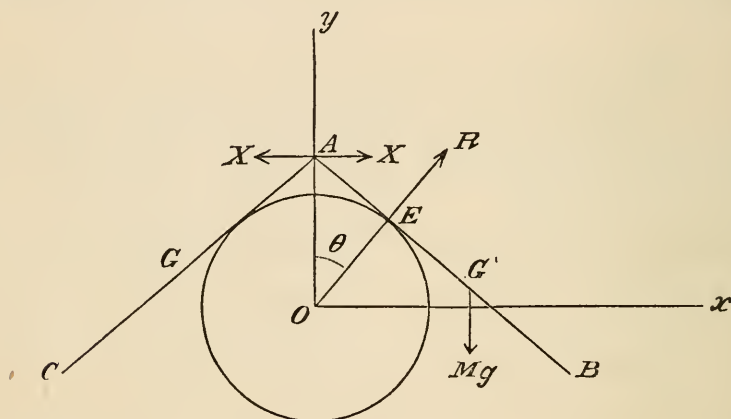
$$a \frac{d^2x}{dt^2} + b \frac{d^2x'}{dt^2} + \dots + \lambda x + \mu x' + \dots = 0,$$

where x, x', \dots are the increments of the independent quantities which determine the position of the body. If the solutions of these equations, of which the number is the same as that of the quantities to be determined, can be expressed in the form of sines or cosines of multiples of t without exponentials, then the motion is oscillatory, and the position of equilibrium is a stable one. If on the other hand the solution involves exponential functions of t , the values of x, x', \dots may some, or all, of them increase indefinitely with t , and our supposition that they are small will be erroneous.

In this case the motion is not oscillatory and the equilibrium is unstable.

118. A single example will suffice.

Two equal heavy rods AB, AC , each of mass M , hinged at A rest symmetrically in a vertical plane over a smooth cylinder. It is required to find the position of equilibrium, and the time of a small oscillation if displaced from that position.



Let the length of either rod be $2a$, and let c be the radius of the cylinder. Let a horizontal and vertical line through O , the centre of the cylinder, in the plane of the rods be taken as axes; let x, y be the co-ordinates of the centre of inertia of AB , and θ the inclination of AB to the horizon. Let R be the pressure of the cylinder on AB , and X the action of AC on AB which from considerations of symmetry we shall assume to be horizontal. Then for the motion of AB , by Art. 79, we have

$$M \frac{d^2 x}{dt^2} = X + R \sin \theta \dots\dots\dots(1),$$

$$M \frac{d^2 y}{dt^2} = R \cos \theta - Mg \dots\dots\dots(2),$$

$$Mk^2 \frac{d^2 \theta}{dt^2} = R \cdot GE + X a \sin \theta \dots\dots\dots(3).$$

The geometrical conditions give

$$x = a \cos \theta \dots\dots\dots (4),$$

$$y = \frac{c}{\cos \theta} - a \sin \theta \dots\dots\dots (5),$$

$$GE = a - c \tan \theta \dots\dots\dots (6).$$

The position of equilibrium is obtained by equating

$$\frac{d^2x}{dt^2}, \quad \frac{d^2y}{dt^2} \text{ and } \frac{d^2\theta}{dt^2}$$

to zero. We thus obtain, if α be the value of θ in this position, eliminating R and X between (1) and (3), and substituting for GE its value from (6),

$$\frac{a - c \tan \alpha}{a \sin \alpha} = \sin \alpha.$$

Therefore $a(1 - \sin^2 \alpha) = c \tan \alpha,$

or $a \cos^3 \alpha = c \sin \alpha \dots\dots\dots (7).$

119. If the rods be slightly displaced from this position we may suppose that $\theta = \alpha + \phi$, where ϕ is a small quantity whose square and higher powers may be neglected.

We have then from (4) and (5),

$$x = a \cos (\alpha + \phi) = a \cos \alpha - a \sin \alpha \cdot \phi,$$

by Taylor's Theorem,

$$\begin{aligned} y &= \frac{c}{\cos (\alpha + \phi)} - a \sin (\alpha + \phi) = \frac{c}{\cos \alpha} - a \sin \alpha \\ &\quad + \phi \left(\frac{c \sin \alpha}{\cos^2 \alpha} - a \cos \alpha \right) \\ &= \frac{c}{\cos \alpha} - a \sin \alpha, \text{ by (7).} \end{aligned}$$

Hence, to the order of approximation required, we have

$$\frac{d^2x}{dt^2} = -a \sin \alpha \cdot \frac{d^2\phi}{dt^2}, \quad \frac{d^2y}{dt^2} = 0, \quad \frac{d^2\theta}{dt^2} = \frac{d^2\phi}{dt^2}.$$

Multiplying (1) by $a \sin \theta$ and subtracting from (3), X disappears, and we obtain

$$M \left(k^2 \frac{d^2 \theta}{dt^2} - a \sin \theta \frac{d^2 x}{dt^2} \right) = R (a - c \tan \theta - a \sin^2 \theta) \\ = R (a \cos^2 \theta - c \tan \theta),$$

or substituting the approximate values

$$M (k^2 + a^2 \sin^2 \alpha) \frac{d^2 \phi}{dt^2} = R (a \cos^2 \alpha - c \tan \alpha) \\ - R \left(2a \sin \alpha \cos \alpha + \frac{c}{\cos^2 \alpha} \right) \phi \\ = -R \left(2a \sin \alpha \cos \alpha + \frac{a \cos \alpha}{\sin \alpha} \right) \phi \dots (8) \text{ by (7).}$$

Also (2) gives

$$0 = R \cos \alpha - Mg - R \sin \alpha \cdot \phi.$$

Hence

$$R = \frac{Mg}{\cos \alpha - \phi \sin \alpha} = \frac{Mg}{\cos \alpha} \{1 + \phi \tan \alpha\},$$

approximately.

Substituting for R in (8), and neglecting squares and higher powers of ϕ , we obtain

$$M (k^2 + a^2 \sin^2 \alpha) \frac{d^2 \phi}{dt^2} = -Mg \alpha \left(2 \sin \alpha + \frac{1}{\sin \alpha} \right) \cdot \phi.$$

$$\text{Therefore } \frac{d^2 \phi}{dt^2} + \frac{ag (2 \sin^2 \alpha + 1)}{\sin \alpha (k^2 + a^2 \sin^2 \alpha)} \cdot \phi = 0 \dots (9).$$

Thus the motion is oscillatory and the time of a complete small oscillation is

$$2\pi \sqrt{\frac{\sin \alpha (k^2 + a^2 \sin^2 \alpha)}{ag (2 \sin^2 \alpha + 1)}},$$

where α is determined in terms of a and c by (7).

120. In the preceding problem there is only one independent *co-ordinate* to determine the position of the system. In such a case another method of arriving at the result is available.

By multiplying (1) by $\frac{dx}{dt}$, (2) by $\frac{dy}{dt}$, (3) by $\frac{d\theta}{dt}$, and adding we shall find that the coefficients of X and R on the right hand side vanish in virtue of (4), (5) and (6). Hence integrating, we obtain the equation of energy, namely

$$M \left\{ \left(\frac{dx}{dt} \right)^2 + \left(\frac{dy}{dt} \right)^2 + k^2 \left(\frac{d\theta}{dt} \right)^2 \right\} = C - 2Mgy \dots (10).$$

and substituting in this for $\frac{dx}{dt}$, $\frac{dy}{dt}$ and y their values in terms of θ we obtain

$$\begin{aligned} \left\{ a^2 \sin^2 \theta + \left(\frac{c \sin \theta}{\cos^2 \theta} - a \cos \theta \right)^2 + k^2 \right\} \left(\frac{d\theta}{dt} \right)^2 \\ = \frac{C}{M} - 2g \left(\frac{c}{\cos \theta} - a \sin \theta \right). \end{aligned}$$

If we denote the co-efficient of $\left(\frac{d\theta}{dt} \right)^2$ by the symbol A and differentiate both sides of the equation with respect to θ we obtain

$$\frac{dA}{d\theta} \left(\frac{d\theta}{dt} \right)^2 + 2A \frac{d^2\theta}{dt^2} = -2g \left(\frac{c \sin \theta}{\cos^2 \theta} - a \cos \theta \right).$$

If we now suppose θ to have a value $\alpha + \phi$ where α is the same as before, and neglect squares and higher powers of ϕ and consequently of $\frac{d\phi}{dt}$, this equation gives, expanding the right hand member by Taylor's Theorem,

$$\begin{aligned} (a^2 \sin^2 \alpha + k^2) \frac{d^2\theta}{dt^2} &= -g\phi \cdot \left\{ \frac{c(1 + \sin^2 \alpha)}{\cos^3 \alpha} + a \sin \alpha \right\} \\ &= -ga \cdot \phi \left(\frac{1 + \sin^2 \alpha}{\sin \alpha} + \sin \alpha \right) \\ &= -ga \cdot \phi \frac{2 \sin^2 \alpha + 1}{\sin \alpha} \end{aligned}$$

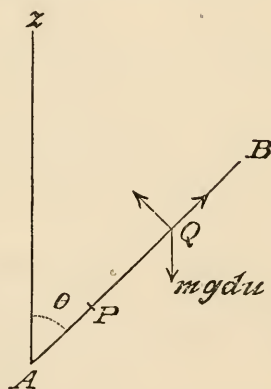
the same equation as before.

121. The general theory of the small oscillations of a system of bodies when their positions depend on a large number of independent quantities, which involves the investi-

gation of the conditions of the stability or instability of equilibrium, is beyond the scope of this treatise. The foregoing example will give the student an idea of the method to be employed in the simple cases that are likely to come before him.

122. It is sometimes required to investigate the *tendency to break* at any point of a rod or wire in motion. We assume that the student is familiar with the statical theorem that the tendency to break at any point of such a rod in equilibrium, is measured by the moment of all the forces which act on the rod on either side of the point. When the rod is in motion we must introduce in addition to the impressed forces the reversed effective forces which, by D'Alembert's principle, form with the impressed forces a system in equilibrium. The tendency to break at any point will then be measured by the sum of the moments of all the impressed and reversed effective forces acting at points on one side of the point considered.

As an illustration we may take the following problem.



A uniform rod AB moveable about one end A , falls from rest in a vertical position: it is required to find the tendency to break at any point P of the rod when it is inclined at an angle θ to the vertical.

By the equation (6) of Art. 52, we have

$$M(a^2 + k^2) \frac{d^2\theta}{dt^2} = Mga \sin \theta,$$

if $2a$ be the length of the rod, and k its radius of gyration about its middle point.

Also if Q be any point of the rod further from A than P , and $AQ = u$, the effective forces on an element of the rod of length du , are $mdu \cdot u \left(\frac{d\theta}{dt} \right)^2$ along QA , and $mdu \cdot u \frac{d^2\theta}{dt^2}$ perpendicular to AB in the direction of θ increasing, m being the mass of a unit of length of the rod. The former of these reversed has no moment round P ; the moment of the latter reversed round P is $mdu \cdot u \frac{d^2\theta}{dt^2} (u - z)$, if $AP = z$.

The moment of the weight of this same element round P is

$$mgdu \cdot (u - z) \sin \theta.$$

Hence the whole tendency to break at P is measured by

$$\begin{aligned} & m \int_z^{2a} \left\{ (u^2 - uz) \frac{d^2\theta}{dt^2} - g(u - z) \sin \theta \right\} du \\ &= m \frac{d^2\theta}{dt^2} \left\{ \frac{8a^3 - z^3}{3} - \frac{4a^2 - z^2}{2} z \right\} - mg \sin \theta \left\{ \frac{4a^2 - z^2}{2} - (2a - z)z \right\}, \\ &= mg \sin \theta \left\{ \frac{a(16a^3 - 12a^2z + z^3)}{6(a^2 + k^2)} - \frac{4a^2 - 4az + z^2}{2} \right\} \\ &= mg \sin \theta \left\{ \frac{16a^3 - 12a^2z + z^3}{8a} - \frac{4a^2 - 4az + z^2}{2} \right\}, \end{aligned}$$

since $k^2 = \frac{a^2}{3}$, by Art. 41,

$$\begin{aligned} &= mg \sin \theta \frac{4a^2z - 4az^2 + z^3}{8a} \\ &= Mg \sin \theta \frac{z(2a - z)^2}{16a^2}. \end{aligned}$$

It may be noticed that this expression vanishes, as of course it ought to do, at each end of the rod, and has its greatest value at a point for which z is $\frac{2a}{3}$.

EXAMPLES. CHAPTER VIII.

1. A horizontal rod of mass m and length $2a$ hangs by two parallel strings of length $2a$ attached to its ends: an angular velocity ω being suddenly communicated to it about a vertical axis through its centre, show that the initial increase of tension of either string equals $\frac{ma\omega^2}{4}$, and that the rod will rise through a space $\frac{a^2\omega^2}{6g}$.

2. A parabolic lamina, cut off by a chord perpendicular to its axis, is kept at rest in a horizontal position by three vertical strings fastened to the vertex and the two extremities of the chord; if the string which is fastened to the vertex be cut, the tension of the others is suddenly decreased one-half.

3. A uniform square lamina is supported in a horizontal position by strings of equal length attached to opposite extremities of a diameter; if one of the strings be cut, determine the instantaneous change of tension of the other.

4. Two equal uniform rods, of length $2a$, are joined together by a hinge at one extremity, their other extremities being connected by an inextensible string of length $2l$. The system rests upon two smooth pegs in the same horizontal line, distant $2c$ from each other. If the string be cut, prove that the initial angular acceleration of either rod will be

$$g \frac{8a^2c - l^3}{\frac{8a^2l^2}{3} + \frac{32a^4c^2}{l^2} - 8a^2cl}.$$

5. An equilateral triangle formed of three equal heavy uniform rods of length a , hinged together at their extremities, is held in a vertical plane with one side horizontal and the vertex downwards. If after falling through any height, the middle point of the upper rod be suddenly stopped, the impulsive strains on the upper and lower hinges will be in the ratio of $\sqrt{13}$ to 1.

If the lower hinge would just break if the system fell through a height $\frac{8a}{\sqrt{3}}$, prove that if the system fell through a height $\frac{32a}{\sqrt{3}}$, the lower rods would just swing through two right angles.

6. The upper extremity of a uniform beam, of length $2a$, is constrained to slide along a smooth horizontal rod without inertia, and the lower along a smooth vertical rod, through the upper extremity of which the horizontal rod passes; the system rotates freely about the vertical rod; prove that, if α be the inclination of the beam to the vertical when in a position of relative equilibrium, the angular velocity of the system will be $\left(\frac{3g}{4a \cos \alpha}\right)^{\frac{1}{2}}$; and, if the beam be slightly displaced from this position, show that it will make small oscillations in the time

$$\frac{4\pi}{\left\{\frac{3g}{a} (\sec \alpha + 3 \cos \alpha)\right\}^{\frac{1}{2}}}.$$

7. A heavy uniform rod AB has its lower extremity A fixed to a vertical axis, and an elastic string connects B to another point C in the axis, such that $AC = \frac{AB}{\sqrt{2}}$; the whole is made to revolve round AC , with such angular velocity that the string is double its natural length and horizontal when the system is in relative equilibrium, and then left to itself; if the rod be slightly displaced in a vertical plane, find the time of a vertical oscillation, the weight of the rod being sufficient to stretch the string to twice its length.

8. A rod, which is the diameter of a circle, is capable of rotating in the plane of the circle about the centre; every particle of an arc of the circle subtending an angle 4α at the centre repels every particle of the rod with a force varying inversely as the square of the distance: if the rod be slightly

displaced from its position of stable equilibrium, prove that the time of a small oscillation, for different values of α , varies as $\left(\frac{\cos 2\alpha}{\sin \alpha}\right)^{\frac{1}{2}}$.

9. The middle point of a uniform rod is fixed midway between two centres of force, which attract with a force varying inversely as the square of the distance. Prove that the time of a small oscillation is

$$\pi (a^2 \sim c^2) \sqrt{\frac{M}{3\mu ac}},$$

where M is the mass of the rod, $2c$ its length, $2a$ the distance between the centres of force, and $\frac{\mu \delta x}{r^2}$ the attraction on an element δx of the rod at a distance r .

10. The extremities of a uniform heavy rod of length $2a$ slide upon two smooth wires which form the upper sides of a square whose diagonal is vertical: prove that the time of a small oscillation is

$$2\pi \sqrt{\frac{4a}{3g}}.$$

Find the greatest angular velocity with which the square may be constrained to move about its vertical diagonal, without destroying the stability of the relative equilibrium of the rod when horizontal.

11. A rough cylinder of radius a loaded so that its centre of gravity is at a distance h from its axis is placed on a board of n times its mass, which can move on a smooth horizontal plane. Find the time of an oscillation when the system is slightly disturbed from its position of stable equilibrium, and prove that if l be the length of the simple equivalent pendulum

$$lh = k^2 + \frac{n}{n+1} (a-h)^2,$$

where k is the radius of gyration of the cylinder about a horizontal axis through its centre of gravity.

12. An elastic string has its ends fastened to the ends of a rod of equal length. The middle point of the string is fastened, and at that point is placed a centre of force which repels every particle of the rod with a force $= \frac{\mu}{(\text{dist.})^2}$. The rod is then moved parallel to itself through a distance equal to half its length. If in this position the elasticity of the string be such that the rod is in equilibrium, show that if slightly displaced perpendicular to its length, the time of a small oscillation will be

$$4\pi \sqrt{\frac{a^3}{\mu (5 + \sqrt{2})}}.$$

13. A rod of length $2a$ and mass M is suspended by a weightless string of length $2l$ over two smooth pegs in the same horizontal line, whose distance apart is $2b$, b being $< a$. When at rest in a horizontal position, it receives a blow Mv at one end in the direction of its length. Show that the initial velocity of the middle point of the string is $v \frac{a-b}{l-b}$.

14. A rod AB , of length $2a$, is capable of motion in a vertical plane round its centre O which is fixed. P and Q are two points vertically above and below O and distant b from it. Two similar elastic strings of equal natural length are fastened at P and Q and also to the end A of the rod. If the rod be pulled out of its position of equilibrium, and then let go, find the angular velocity in any subsequent position, supposing the string to remain stretched all through the motion, and show that the time of a small oscillation is $\pi \sqrt{\frac{m(a^2 + b^2)^{\frac{3}{2}}}{3\lambda b^2}}$, where m is the mass of the rod and λ the coefficient of elasticity.

15. A wire in the form of the portion of the curve $r = a(1 + \cos \theta)$ cut off by the initial line rotates about the origin with angular velocity ω , show that the tendency to break at a point $\theta = \frac{\pi}{2}$ is measured by $\frac{m \cdot 12 \sqrt{2}}{5} \omega^2 a^3$, where m is the mass of a unit of length.

16. A wire is bent into a circular form and is placed with the diameter through the crack A vertical, and the other extremity B of this diameter is fixed: it is then made to rotate with an angular velocity ω about AB . Find the tendency to break at any point.

17. A system consisting of two uniform rods AC , CB rigidly connected together at C , and at right angles, is whirled away in any manner on an infinite horizontal smooth plane, so that every point always touches the plane; show that the tendency to break at any moment at C is proportional to

$$\frac{CA^2 \cdot CB^2}{CA + CB}.$$

18. The rigid body described in question 17 falls without rotation and strikes a smooth horizontal plane at B ; if there is no rotation produced by the impact, show that the inclination of BC to the horizon is $\tan^{-1} \frac{BC(2AC + BC)}{AC^2}$.

Find in that case the impulsive breaking strain at C .

19. A circular wire is revolving uniformly about its centre fixed. If it be cracked at any point, show that the tendency to break at an angular distance α from the crack is proportional to $\sin^2 \frac{\alpha}{2}$.

ANSWERS TO EXAMPLES.

CHAPTER I.

1. $\omega\sqrt{14}$ round the line $6x = -3y = 2z$.
2. Use Art. 17.
4. By example 2: by changing the rotation to one round a parallel axis properly chosen the velocity perpendicular to the axis of rotation can be destroyed.
5. The fixed point and line are in each case the focus and directrix of the parabola.
6. The focus of the parabola at each instant is the instantaneous centre of rotation, the line itself is the tangent at the vertex.
7. $a\omega$ parallel to one edge, a being the length of an edge.
8.
$$l \tan^{-1} \frac{k}{x} + m \tan^{-1} \frac{k}{y} + n \tan^{-1} \frac{k}{z} = 0,$$

where
$$k = \sqrt{yz + zx + xy}.$$
9. Use last paragraph of Art. 3. Yes,—round a line perpendicular to each of the two straight lines.

CHAPTER II.

1. (a) The acceleration of the chain

$$= \left\{ \frac{x}{l} (\sin \alpha + \sin \beta) - \sin \beta \right\} g,$$

where x is the length on the plane inclined at an angle α to the horizon, and l its whole length. Equating this to $\frac{d^2x}{dt^2}$ and integrating, the motion can be determined.

(γ) The motion is given by an equation

$$8a \frac{d^2x}{dt^2} = g \left(2x - \frac{x^2}{8a} \right).$$

2. Backwards, for the centre of gravity must descend vertically.

3. When the oarsmen move their bodies the boat must move in an opposite direction. The water opposes less resistance to a sudden rapid motion of the boat through it than to a slow one.

4. Solve as a statical problem, applying to each element of the rod a force $m\delta s \cdot r\omega^2$ perpendicular to the axis of rotation.

CHAPTER III.

1. $\frac{1}{12} Mab.$

2. $M \frac{a^2}{4}; \quad M \frac{a^2 + b^2}{4}.$

3. (1) $M \frac{r_1^2 + r_1 r_2 + r_2^2}{3};$ r_1, r_2 being the radii vectores of the extremities. (2) $M \frac{88a^2}{245}.$

4. (1) $M \frac{r_1^3 - r_2^3}{r_1 - r_2}.$ (2) $\frac{38}{175} Ma^2.$ (3) $M \frac{b^2 + c^2}{5}.$

7. (1) $\frac{3Mh^2 \tan^2 \alpha}{10}.$ (2) $\frac{3M}{5} h^2 (1 + \frac{1}{4} \tan^2 \alpha).$

(3) $\frac{3Mh^2}{10} \sin^2 \alpha (3 + \frac{1}{2} \tan^2 \alpha).$

8. $\frac{Mh^2}{3} \sin^2 \alpha (3 + \frac{1}{2} \tan^2 \alpha).$

9. $M \frac{b^2 + c^2}{4},$ the density being taken to be always the numerical value of the given expression; otherwise zero. 10. $\frac{7a^2}{16}.$

11. The solid is produced by the revolution of the curve $r = a(1 + \cos \theta)$ round the initial line.

$$(1) \quad M \cdot \frac{24a^2}{35}. \quad (2) \quad M \cdot \frac{160a^2}{143}.$$

12. Deduce from Art. 46 by similar triangles.

13. Divide the tetrahedron into triangular slices by planes parallel to the face opposite the given angle. To each of these slices apply the result of the last question. If h_1, h_2, h_3 be the perpendicular distances of the other three angular points from the given plane, the required moment

$$= \frac{M}{10} \{h_1^2 + h_2^2 + h_3^2 + h_1h_2 + h_2h_3 + h_3h_1\}.$$

14. Deduce from 13 by the help of Arts. 32 and 34.

15. Take the given edge as axis of z , any point in it as origin, and the plane through it and the middle point of the opposite edge as plane of zx . The required condition is that the two opposite edges are mutually perpendicular. Use Example 14.

16. Use Example 14.

17. One axis will be the tangent to the base at the point required. By Example 7 calculate the principal moments at the centre of inertia, and by Art. 37 obtain the equation of the momental ellipsoid at the point required.

18 and 19. Use Art. 37.

$$21. \quad \frac{2Mab}{5\pi}.$$

22. Use Art. 37.

CHAPTER IV.

2. The distance of each axis from the parallel one through the centre of inertia can be determined in terms of the required radius of gyration.

4. The equation of motion is

$$(k^2 + a^2) \frac{d^2\theta}{dt^2} = -ga \sin \theta - ng \cdot \frac{hc}{h-c} \sin \theta + \frac{nghc \sin \theta}{\sqrt{h^2 + c^2 - 2hc \cos \theta}}.$$

Time of small oscillation $= 2\pi \sqrt{\frac{a^2 + k^2}{ag}}$, since the terms with n when expanded do not contain a lower power than θ^2 .

5. The angular velocity of the rod and the linear velocity of the ball are reversed at each impact.

6. Take T_1, T_2 as the tensions of the string at the two ends of the horizontal diameter and write down the equations of motion of the plate and the two portions of string separately. Eliminate T_1 and T_2 and integrate.

7. The greatest angular velocity is produced when the tangential action is just μ times the normal action during each part of the impact.

$$8. \sqrt{\frac{12g\sqrt{2}}{7b}};$$

$$\text{impulsive tension} = \frac{M}{2} b \cos \frac{\theta}{2} \sqrt{\frac{6g \left(\sqrt{2 - 2 \sin \frac{\theta}{2}} \right)}{b \left(2 + 3 \cos^2 \frac{\theta}{2} \right)}},$$

θ being the angle between the door and doorway when stopped. For the other results proceed as in Art. 59.

$$9. \frac{2}{\sqrt{5}}.$$

10. By Art. 55 the co-tangent of the angle required, in any position,

$$= \frac{(k^2 + 3h^2) \cot \theta + h^2 \tan \theta}{3h^2}.$$

12. The motion is really one of rotation about the centre of the ring.

CHAPTER V.

1. Use Art. 64, remembering that $\omega_y = \omega_z = 0$; $A' = B' = C' = 0$, and $B = M \cdot \frac{a^2 + c^2}{5}$.

2. If ϕ be the angle which the plane through the instantaneous axis and the principal axis of C makes with the plane of AC ,

$$\tan \phi = \frac{\omega_2}{\omega_1},$$

therefore
$$\frac{d\phi}{dt} = \cos^2 \phi \cdot \frac{d\left(\frac{\omega_2}{\omega_1}\right)}{dt} = \frac{\omega_1 \frac{d\omega_2}{dt} - \omega_2 \frac{d\omega_1}{dt}}{\omega_1^2 + \omega_2^2};$$

and use Art. 68.

3. Differentiate $\Sigma m(y^2 + z^2)$, &c., and use the formulæ of Art. 7. The angular momenta are given in Art. 65.

4. Take the equations of Art. 68, putting $A = B$: we easily get from the data

$$L = \frac{k\omega_1}{\sqrt{\omega_1^2 + \omega_2^2}}, \quad M = \frac{k\omega_2}{\sqrt{\omega_1^2 + \omega_2^2}}, \quad N = \frac{k\omega_3}{\sqrt{\omega_1^2 + \omega_2^2}},$$

where k is some constant. The equations give the values of $\sqrt{\omega_1^2 + \omega_2^2}$ and ω_3 ; $\sqrt{\omega_1^2 + \omega_2^2} = \frac{k}{A} \cdot t$.

5. From Art. 68 we get, by integration and having regard to initial circumstances,

$$B(A - B)\omega_2^2 - C(C - A)\omega_3^2 = 0,$$

whence the result follows.

6. Use the equations (1), (2), (3) of Art. 71, remembering that $C = A + B$.

7. Take C as the mean axis, and use the equations of Arts. 74 and 71.

Since $kh^4 = Ch^2$ it easily follows that the ratio of ω_1^2 to ω_2^2 is constant; whence ψ is constant by (1) of Art. 74, and therefore.

$\frac{d\phi}{dt}$ also by (3). And by integrating (2) the result can be obtained. To give the result in the question, C and B must be interchanged and G written for k^2 .

CHAPTER VI.

1. The blow must be applied at the point which would be the centre of percussion if the axis were fixed.

2. If x be the distance of the particle from *fixed* axis,

$$\omega' = \frac{A\omega}{A + Mx^2}, \quad v = \frac{A\omega x}{A + Mx^2},$$

where M is mass of particle, v its velocity after impact, and ω, ω' the angular velocities before and after impact; therefore v is least when $Mx^2 = A$, A being the moment of inertia of the lamina about the fixed axis.

3. The inclination of the rod to the horizon is given by

$$Ma \sin \theta = M' \{ \sqrt{c^2 + 4a^2 - 4ac \cos \theta} - c + 2a \},$$

where M, M' are the masses of the rod and weight, $2a$ the length of the rod, and c the distance of the pulley from the hinge.

4. Use Art. 80.

6. If ϕ, θ be the inclinations to the vertical of the rod and the radius to the point at which the end of the rod is fastened, they satisfy the conditions

$$c \sin \theta = 2a \sin \phi \quad \text{and} \quad c^2 \left(\frac{M}{2} + M' \sin^2 \theta \right) \left(\frac{d\theta}{dt} \right)^2 + \frac{4M'a^2}{3} \left(\frac{d\theta}{dt} \right)^2 \\ + 2M'ac \sin \theta \sin \phi \frac{d\theta}{dt} \frac{d\phi}{dt} = C - 2M'g(c \cos \theta + a \cos \phi).$$

7. Write down the equations of motion of the disc and each rod separately.

9. Take the acceleration of the centre of the ring along the tangent and normal to its path. If V, v be the velocities of this point at first and after a time t , ω the angular velocity after time t , and ϕ the angle between the initial normal and that at time t , we easily get $v + a\omega = V$, $V = v\epsilon^{\mu\phi}$, as long as there is sliding. When perfect rolling begins $v = a\omega$.

10. See Art. 81.

11. Equations (1) and (2) of Art. 81 apply to the sphere. Equations (3) require $-y\omega$ and $+x\omega$ for zero on the right-hand side, ω being the angular velocity of the disc at the time. For the motion of the disc we have

$$mk'^2 \frac{d\omega}{dt} = F_x \cdot y - F_y \cdot x.$$

Whence can be obtained

$$mk'^2 \omega + M \left(x \frac{dy}{dt} - y \frac{dx}{dt} \right) = \text{constant} = mk'^2 \Omega,$$

$$\frac{dx}{dt} (a^2 + k^2) + k^2 y \omega = 0, \quad \frac{dy}{dt} (a^2 + k^2) - k^2 x \omega = 0,$$

and the result follows.

12. See Arts. 85, 86 and also 95, 96 for the value of the vis viva.

13. Take the equations of Articles 85 and 86 and suppose an additional friction $= eF_y$. If v'' be the resulting velocity parallel to the wall

$$v'' = v - (1 + e) \frac{k^2 (v + a\omega)}{a^2 + k^2} = v - \frac{2}{3} (1 + e) (v + a\omega),$$

and if the ball retraces its path, this $= -ev$, since $u'' = -eu$.

Hence $2a\omega = 5v$, which is $5V \cos \alpha$.

14. Additional frictions F_1, F_2 , parallel to Ox and Oy , must be introduced into the equations of Art. 85 at the point of contact of the ball with the table. The problem must first be solved for an inelastic ball. We finally get

$$F_y = -\frac{7}{5} M (v + a\omega), \quad F_x = \frac{5}{4} \frac{3}{5} M (v + a\omega) :$$

whence

$$v'' = v - (1 + e) \cdot \frac{2}{9} (v + a\omega),$$

and equating this to $-ev$ the result follows.

15. If x, y, z be the co-ordinates of the point of application of the blow, X, Y, Z the components of the blow, the general condition for rolling is that

$$(X^2 + Y^2) \left(\frac{az}{k^2} - 1 \right)^2 - \frac{2aZ}{k^2} \left(\frac{az}{k^2} - 1 \right) (Xx + Yy) \\ + Z^2 \left\{ \frac{a^2 (a^2 - z^2)}{k^4} - \mu^2 \left(1 + \frac{a^2}{k^2} \right) \right\} \neq 0,$$

which, putting

$$\mu = \sqrt{\frac{3}{7}} \text{ and } k^2 = \frac{2a^2}{5},$$

reduces to

$$\left(\frac{5z}{2a} - 1\right) \left\{ (X^2 + Y^2) \left(\frac{5z}{2a} - 1\right) - 5aZ(Xx + Yy) - Z^2 \left(1 + \frac{5z}{2a}\right) \right\} \neq 0.$$

16. The motion of the centre of the sphere is all in one plane. The condition is that $R = 0$.

17. The condition is that F becomes equal to μR . The equations for the cylinder and particle must be written down separately.

CHAPTER VII.

1. $\frac{a\Omega \sqrt{(c^2 + a^2)(3a^2 - c^2)}}{2a^2}$, Ω being the original angular velocity, and c the original distance of the particle from the fixed end of the tube.

2. $2a\Omega\sqrt{7}$, if Ω be the angular velocity, after the impulse between the rod and tube at starting has taken place.

$\Omega = \frac{\Omega'}{14}$ if Ω' be the angular velocity of the tube before this impulse.

3. The position of the insect, while on the disc, is determined by

$$r = Vt, \quad \frac{d\theta}{dt} = \frac{Mk^2\Omega}{Mk^2 + mr^2}.$$

4. The equation of conservation of moment of momentum gives

$$(Mk^2 + mr^2) \frac{d\phi}{dt} + mr^2 a = \text{constant},$$

where ϕ is the angle turned through by the disc.

Also

$$r^2 = a^2 \cos 2at.$$

6. The angular velocity of each rod after the impulse = $\frac{3v}{2a}$.

7. The moment of momentum and the energy of the system remain unaltered. When the angle between the rods is greatest $\frac{d\theta}{dt} = \frac{d\phi}{dt}$, with a notation like that of Art. 105, which see.

8. Compare Arts. 105 and 108.

9. The energy and moment of momentum of the system are each constant.

10. Compare 105 and Example 7.

11. Compare 105 and 108.

12. Compare 105 and Example 7.

14. The blow will always pass through the instantaneous centre of rotation. See Art. 3.

15. If θ, ϕ be the inclination of the rod to the vertical, and of the vertical plane through the rod to some fixed plane, the energy of the rod

$$= \frac{1}{2} M \frac{4a^2}{3} \left\{ \left(\frac{d\theta}{dt} \right)^2 + \sin^2 \theta \left(\frac{d\phi}{dt} \right)^2 \right\},$$

and the moment of its momentum round a vertical axis

$$= M \frac{4a^2}{3} \sin^2 \theta \frac{d\phi}{dt}.$$

The latter is constant and the former $= C + Mga \cos \theta$. When the rod makes the least angle with the vertical $\frac{d\theta}{dt} = 0$.

16. Solve as 15 and find when $\frac{d\theta}{dt}$ is a maximum.

17. Use Art. 64 to calculate the angular velocities produced, and Art. 96 to calculate the energy.

18. The equations of motion of each rod are easily written down. The last result follows from the fact that the centre of inertia of the whole has no motion horizontally.

CHAPTER VIII.

1. The centre of the rod rises vertically. There are thus two equations of motion, each involving T . The equation of energy will give the second result.

2. Solve as in Art. 115.

3. T becomes $\frac{Mg}{7}$.

4. Treat as Art. 115.

5. If X' , Y' be the horizontal and vertical components of the action at the upper hinge and X the action at the lower, which is evidently horizontal, we easily get

$$X + X' = 0, \quad Mv = Y', \quad (X - X') \frac{a\sqrt{3}}{2} - Y' \frac{a}{2} = 0,$$

whence the first result follows.

7. The equations of motion are

$$\sin^2 \theta \frac{d\phi}{dt} = \text{constant} = \frac{\omega}{2},$$

where θ and ϕ are the usual angular co-ordinates of the rod, and

$$k^2 \left\{ \sin^2 \theta \left(\frac{d\phi}{dt} \right)^2 + \left(\frac{d\theta}{dt} \right)^2 \right\} = C - 2ga \cos \theta - \frac{g\sqrt{2}}{a} \left(2a \sin \theta - \frac{a}{\sqrt{2}} \right)^2;$$

whence

$$k^2 \left(\frac{d\theta}{dt} \right)^2 = C - 2ga \cos \theta - \frac{g\sqrt{2}}{a} \left(2a \sin \theta - \frac{a}{\sqrt{2}} \right)^2 - \frac{k^2 \omega^2}{4 \sin^2 \theta};$$

and by differentiating and putting $\theta = \frac{\pi}{4} + \psi$ and then expanding the right-hand side to the first power of ψ so as to get an equation of the form $k^2 \frac{d^2 \psi}{dt^2} = A - B\psi$, the condition of relative equilibrium,

which requires that $A = 0$, gives $\omega^2 = \frac{3g}{2a\sqrt{2}}$, and the time of a small oscillation

$$= 2\pi \sqrt{\frac{k^2}{B}} = 2\pi \sqrt{\frac{2a\sqrt{2}}{15g}}.$$

The given geometrical condition shows that $\theta = \frac{\pi}{4}$ in the position of relative equilibrium.

8. The position of stable equilibrium is when the diameter is perpendicular to the line bisecting the arc. The moment of the repulsive forces on the rod round the centre when the rod is displaced by an angle θ from this position is easily found to be

$$= \sqrt{2}\mu a \int_{-2a}^{2a} \frac{\sin \frac{\theta + \phi}{2} d\phi}{\cos(\theta + \phi)};$$

the general integral of this

$$= \mu \log \frac{\sqrt{2} \cos \frac{\theta + \phi}{2} + 1}{\sqrt{2} \cos \frac{\theta + \phi}{2} - 1};$$

and taking this between the limits, and expanding in powers of θ , the result follows.

In this and the following question the attraction of a particle on a rod must be remembered to be the same as that of the rod on the particle, and to bisect the angle between the lines joining the ends of the rod to the particle.

10. $\omega^2 < \frac{3g}{2a}$. The equation giving the time of a small oscillation is

$$(a^2 + k^2) \frac{d^2\theta}{dt^2} = -\{ag - (a^2 - k^2) \omega^2\} \theta,$$

if θ is the inclination of the rod to the horizon.

11. The centre of inertia of the whole system has no horizontal motion. The equation of energy and the geometrical conditions will give the rest.

12. The only motion to be considered is that of the centre of inertia.

13. The initial velocity of the rod is v along AB . Hence the velocity of either end of the string in the direction of its length

$$= v \cdot \frac{a - b}{l - b}.$$

$$14. Mk^2 \left(\frac{d\theta}{dt} \right)^2 = C + \lambda \{ \sqrt{a^2 + b^2 - 2ab \sin \theta} - \sqrt{a^2 + b^2 + 2ab \sin \theta} \}.$$

15. The tendency to break

$$= m \int_0^{\frac{\pi}{2}} r \omega^2 \cdot \frac{ds}{d\theta} \cdot a \cos \theta \cdot d\theta.$$

16. The tendency to break at a point at an angular distance α from B is $2m\alpha^3\omega^2 \cdot \cos^4 \frac{\alpha}{2}.$

17. The only effective forces reversed which have to be considered are forces $m r \omega^2$ on each particle from the centre of inertia outwards. The sum of the moments of all these on CA or CB , round C , will give the required result.

18. The centre of inertia must be vertically above B .

The force at C consists of a couple and a vertical force. The latter $= \mu \cdot CA \cdot v$ and the former $= \mu \cdot CA \cdot CB \cdot \sin \theta$, where μ is the mass of a unit of length, v the velocity of the rods before impact, and θ the inclination of CB to the horizon.





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